Uniform asymptotic estimates of transition probabilities on combs

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Abstract. We investigate the asymptotical behaviour of the transition probabilities of the simple random walk on the 2-comb. In particular we obtain space-time uniform asymptotical estimates which show the lack of symmetry of this walk better than local limit estimates. Our results also point out the impossibility of getting Jones-type non-Gaussian estimates.

Keywords: Uniform estimate, uniform Lebesgue theorem, Green function, transition probability, Cauchy integral, comb.

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1. Introduction

Given a random walk $(Z_n)_{n\geq 0}$ on a graph X, there are many related questions regarding its behaviour when the discrete time parameter n goes to infinity. Classical questions of this kind are, for instance: will the random walk visit a given vertex of the graph only a finite number of times (with probability one)? will it leave any bounded set after a finite time (with probability one)? Moreover, if we denote by $p^{(n)}(x,y)$ the n-step probabilities of the random walk from the vertex x to the vertex y of X, we can study some features of the sequence $(p^{(n)}(x,y))_n$, for instance answer to the question: is it asymptotic to some numerical sequence?

Answers to the first two questions are theorems and criteria for recurrence and transience; answers to the latter question are provided by *local limit theorems*, that is theorems which give a numerical estimate of $p^{(n)}(x,y)$ for fixed x, y as n tends to infinity.

The present work studies the asymptotic behaviour of the transition probabilities of the simple random walk on the 2-comb, which is a graph obtained attaching at each point of $\mathbb{Z}$ another copy of $\mathbb{Z}$ by its origin. In other words, this graph is obtained from the usual square grid $\mathbb{Z}^2$ by deleting all edges that are parallel to the x-axis except those on the x-axis itself.

A local limit theorem for the 2-comb (and in general for d-dimensional combs) and $x = y$ is well known: we observe how to extend it to the asymptotic estimate of $(p^{(n)}(x,y))_n$ for any x and y (equation (3.3)). The simple random walk on the 2-comb lacks of symmetry (more precisely it is not isotropic – see Bertacchi and Zucca [1]) and this feature is not shown by local limit estimates. To stress the different behaviour the random walk has in the two principal directions (vertical and horizontal: see Figure 1) one needs space-time estimates.

A *space-time asymptotic estimate* is a result which provides an estimate of $p^{(n)}(x,y)$ as n tends to infinity, uniform with respect to the quotient $d(x,y)/n$ lying in a suitable range. Of course local limit theorems can be derived from space-time estimates. We provide space-time asymptotic estimates for the $p^{(n)}(x,y)$ when $y = o := (0,0)$ and $x = (k,0)$ or $x = (0,k)$, that is results of the form

$$p^{(n)}(x,o) \stackrel{n}{\sim} Cf\left(\frac{d(x,o)}{n}, n\right) n^{-d}$$

where C, d are constants and f is a real valued function which all depend on the range of $d(x,o)/n$. From these results all known limit theorems for the 2-comb can be derived as a particular case.

The technique we exploit is essentially a Laplace-type estimate of integrals, since the transition probabilities can be written as integrals thanks to the Cauchy formula for the coefficients of the power series of an holomorphic function:

$$p^{(n)}(x,o) = \frac{1}{2\pi i}\int_\gamma \frac{G(x,o|z)}{z^{2n+1}}\,\mathrm{d}z$$

where $G(x,o|z) := \sum_{n=0}^{\infty} p^{(n)}(x,o)z^n$ is the Green function associated to the walk and has an explicit expression, and γ is a positively oriented, simple closed curve in $\mathbb{C}$ surrounding 0. The basic idea is that the main part of the integral is given by integration on the part of the curve which is closer to the singularity $z=1$ of $G(x,0|z)$. To develop this idea into mathematical terms we first separate in the integrand the part with algebraic behaviour and the one with exponential behaviour, then we choose a suitable curve of integration and its parametrization. Afterwards, we write the Taylor expansion of the argument of the exponential part of the integrand as a function of the parameter of the curve and we finally show that it is possible to choose a piece of the curve on which integration gives the asymptotic behaviour of the transition probabilities. It is remarkable that the above mentioned Taylor expansions are very different in the two cases $x=(k,0)$ and $x=(0,k)$. This reflects in two different asymptotic behaviours of the transition probabilities when $d(x,o)/n$ tends to 0: in the first case

$$p^{(n)}(x,o) \overset{n}{\sim} c_1 \exp(c_2\, n(d(x,o)/n)^{4/3})\, n^{-3/4},$$

while in the second case

$$p^{(n)}(x,o) \overset{n}{\sim} c_3 \exp(c_4\, n(d(x,o)/n)^{2})\, n^{-3/4}.$$

In particular this shows that for the 2-comb it is impossible to find a uniform estimate involving the spectral dimension and the walk dimension such as the one found by Jones for the 2-dimensional Sierpinski graph (see Paragraph 10).

We give a brief outline of the paper: in Paragraph 2 we listed the definition of uniform estimate with respect to a parameter and two Lebesgue-type theorems which are needed to obtain such estimates. Paragraph 3 recalls local limit theorems and generating functions for combs. Then in Paragraphs 4 and 5 uniform estimates are proved for the vertical direction, while in Paragraphs 6, 7, 8 and 9 we prove uniform estimates for the horizontal direction. In the last paragraph we discuss the obtained results and possible extensions.

2. Asymptotic estimates: definitions and technical results

In this paper we are concerned with asymptotic estimates of transition probabilities. We then give some useful definitions and theorems.

Definition 2.1. *Given two sequences $\big(a_n(x)\big)_n$ and $\big(b_n(x)\big)_n$ of complex functions defined on a space X and a sequence $(A_n)_n$ of subsets of X, we say that a_n is asymptotic to b_n (and we write $a_n \overset{n}{\sim} b_n$) as n tends to infinity, uniformly with respect to $x \in A_n$ if and only if there exists a sequence of complex functions $\big(o_n(x)\big)_n$ and $n_0 \in \mathbb{N}$ such that:*

(i) $a_n(x) = b_n(x)(1+o_n(x))$ for every $n \ge n_0$, $x \in A_n$;

(ii) for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$, $n_\varepsilon \ge n_0$, such that for every $n > n_\varepsilon$, $x \in A_n$ we have $|o_n(x)| < \varepsilon$.

This definition extends the usual one (let X be a one point space and $A_n \equiv X$ for all $n \in \mathbb{N}$).

Definition 2.2. *Let us consider a sequence $\big(a_n(x)\big)_n$ of functions defined on a space X with values on a metric space (Y,d) and let $(A_n)_n$ be a sequence of subsets of X. Let $b : X \to (Y,d)$, we say that $(a_n)_n$ converges to b when n tends to infinity, uniformly with respect to $x \in A_n$ if and only if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \ge n_\varepsilon$, $x \in A_n$ we have $d(a_n(x), b(x)) < \varepsilon$.*

This is an extension of the well-known definition of uniform convergence (usually stated for $A_n = A \subset X$). The following result provides an extension of Lebesgue's bounded convergence theorem to the case of uniform convergence (in the preceding sense).

Theorem 2.3. *Let (X,Σ,μ) be a σ-finite measure space and let $(f_n)_n : (X,\Sigma,\mu)\times Y \to \mathbb{C}$ be a sequence of complex functions which are measurable with respect to $x \in X$ for every fixed $y \in Y$ and $(A_n)_n \in \mathcal{P}(Y)$ such that $f_n(x,y) \to f(x,y)$ when n tends to infinity uniformly with respect to $x \in A_n$. Then f is measurable with respect to $x \in X$ for every fixed $y \in Y$ and if there exists $g \in L^1(X,\Sigma,\mu)$ such that $|f_n(x,y)| \le g(x)$, μ a.e., for every $n \in \mathbb{N}$ and for every $y \in A_n$, then $f_n(\cdot,y) \to f(\cdot,y)$ in $L^1(X,\Sigma,\mu)$ uniformly with respect to $y \in A_n$.*

Proof. Let us fix $\varepsilon > 0$ and a non decreasing covering of X, $(M_n)_n$ with finite measure measurable sets. From Lebesgue's monotone convergence theorem, if $h \in L^1(X,\Sigma,\mu)$, then there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \ge n_\varepsilon$, we have $\int_X |h|\,\mathrm{d}\mu < \varepsilon$. Moreover, by the absolute continuity of the Lebesgue integral, there exists δ_ε such that if $E \in \Sigma$, $\mu(E) < \delta_\varepsilon$ then $\int_E |h|\,\mathrm{d}\mu < \varepsilon$.

Now let

$$M_m(n) := \left\{ x \in M_m : \forall p \ge n, |f_p(x,y) - f(x,y)| < \frac{\varepsilon}{\mu(M_m)}, \forall y \in A_p \right\},$$

where $m,n \in \mathbb{N}$. It is easy to show that $M_m(n) \supseteq M_m(n-1)$ and $\mu(M_m \setminus \cup_{n\in\mathbb{N}} M_m(n)) = 0$, then if we take $m, n_0 \in \mathbb{N}$ such that $\int_{M_m^c} g\,\mathrm{d}\mu < \varepsilon$ and $\mu(M_m \setminus M_m(n)) < \delta_\varepsilon$, $\forall n \ge n_0$ then

$$\begin{aligned}
&\int_X |f_n(x,y) - f(x,y)|\,\mathrm{d}\mu = \\
&= \int_{M_m^c} |f_n(x,y) - f(x,y)|\,\mathrm{d}\mu + \int_{M_m} |f_n(x,y) - f(x,y)|\,\mathrm{d}\mu \le \\
&\le 2\int_{M_m^c} g(x)\,\mathrm{d}\mu + 2\int_{M_m\setminus M_m(n)} g(x)\,\mathrm{d}\mu + \int_{M_m(n)} |f_n(x,y) - f(x,y)|\,\mathrm{d}\mu < 5\varepsilon
\end{aligned}$$

for all $n \ge n_0$, and for all $y \in A_n$. ∎

Sometimes it will be impossible to exhibit a limit function for our estimates, but we will be able to find a sequence of functions asymptotic to the given one and much simpler. In that direction works the following theorem.

Theorem 2.4. *Let (X,Σ,μ) be a measure space and let $(f_n)_n, (h_n)_n, (o_n)_n : (X,\Sigma,\mu)\times Y \to \mathbb{C}$ be three sequences of functions which are measurable with respect to $x \in X$ for every fixed $y \in Y$. Let $(A_n)_n$ be a sequence of sets in $\mathcal{P}(Y)$ such that $h_n(x,y) = f_n(x,y)(1 + o_n(x,y))$, μ a.e., for all $n \in \mathbb{N}$, for all $y \in A_n$, where $o_n(x,y) \to 0$, when n tends to infinity, μ a.e., uniformly with respect to $y \in A_n$. If there exist $g, g_1 \in L^1(X,\Sigma,\mu)$ such that*

$$|h_n(x,y)| \le g(x), \quad |f_n(x,y)| \le g_1(x)$$

μ a.e., for all $n \in \mathbb{N}$, for all $y \in A_n$ and we can choose $c > 0$ such that $\left|\int_X f_n(x,y)\,\mathrm{d}\mu\right| \ge c$, for all $n \in \mathbb{N}$ and for all $y \in A_n$ then

$$\int_X f_n(x,y)\,\mathrm{d}\mu \overset{n}{\sim} \int_X h_n(x,y)\,\mathrm{d}\mu$$

uniformly with respect to $y \in A_n$.

Proof. Let us fix $\varepsilon > 0$ and define $X_n(\varepsilon) = \{x \in X : |o_m(x,y)| < \varepsilon, \forall m \ge n, \forall y \in A_m\}$ then $X_n(\varepsilon) \supseteq X_{n-1}(\varepsilon)$ and $\mu(X \setminus \cup_{n\in\mathbb{N}} X_n(\varepsilon)) = 0$. For all $n \in \mathbb{N}$ we define

$$\begin{aligned}
a_n(\varepsilon) &:= \int_{X_n^c(\varepsilon)} g\,\mathrm{d}\mu, \\
b_n(\varepsilon) &:= \int_{X_n^c(\varepsilon)} g_1\,\mathrm{d}\mu,
\end{aligned}$$

and it is easy to show that $a_n(\varepsilon) \to 0$, $b_n(\varepsilon) \to 0$ if n tends to infinity. Moreover it is useful to fix

$$\begin{aligned}
&I(y,n) := \textstyle\int_X f_n(x,y)\,\mathrm{d}\mu, \qquad I_1(y,n) := \int_X h_n(x,y)\,\mathrm{d}\mu, \\
&I_2(y,n) := \textstyle\int_{X_n(\varepsilon)} f_n(x,y)\,\mathrm{d}\mu,
\end{aligned}$$

(the thesis is that $I(y,n) \stackrel{n}{\sim} I_1(y,n)$ uniformly with respect to $y \in A_n$); then

$$|I(y,n) - I_1(y,n)| \le |I(y,n) - I_2(y,n)| + |I_1(y,n) - I_2(y,n)|.$$

Now

$$|I(y,n) - I_2(y,n)| \le \int_{X_n^c(\varepsilon)} |f_n(x,y)| \, \mathrm{d}\mu \le b_n(\varepsilon), \quad \forall n, \forall y \in A_n;$$

besides

$$\begin{aligned} |I_1(y,n) - I_2(y,n)| &\le \int_{X_n(\varepsilon)} |f_n(x,y) - h_n(x,y)| \, \mathrm{d}\mu + \int_{X_n^c(\varepsilon)} |h_n(x,y)| \, \mathrm{d}\mu \le \\ &\le \varepsilon \int_{X_n(\varepsilon)} |f_n(x,y)| \, \mathrm{d}\mu + a_n(\varepsilon) \le \\ &\le \varepsilon \|g_1\|_1 + a_n(\varepsilon), \quad \forall n, \forall y \in A_n. \end{aligned}$$

Keeping in mind that $|I(n,y)| \ge c$ we have

$$\begin{aligned} \left| 1 - \frac{I_1(y,n)}{I(y,n)} \right| &\le \frac{|I(y,n) - I_2(y,n)|}{|I(y,n)|} + \frac{|I_1(y,n) - I_2(y,n)|}{|I(y,n)|} \le \\ &\le \frac{1}{c}(b_n(\varepsilon) + \varepsilon \|g_1\|_1) + \frac{1}{c} a_n(\varepsilon), \quad \forall n, \forall y \in A_n. \end{aligned}$$

Now we take n_ε such that for all $n \ge n_\varepsilon$ we have $a_n(\varepsilon) + b_n(\varepsilon) \le 2\varepsilon \|g\|_1$ and we finally obtain

$$\left| 1 - \frac{I_1(y,n)}{I(y,n)} \right| \le 3\varepsilon c^{-1} \|g\|_1, \quad \forall n \ge n_\varepsilon, \forall y \in A_n.$$

□

The last (non-standard) proposition we need deals with the triangular inequality for power series: we are interested in the cases where a strict inequality holds.

Proposition 2.5. *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be a power series with non negative coefficients (and positive radius of convergence): then $|f(z)| \le f(|z|)$. Moreover, if there exists $n \in \mathbb{N}$ such that a_n and a_{n+1} are both strictly positive, then $|f(z)| = f(|z|)$ if and only if $z = |z|$.*

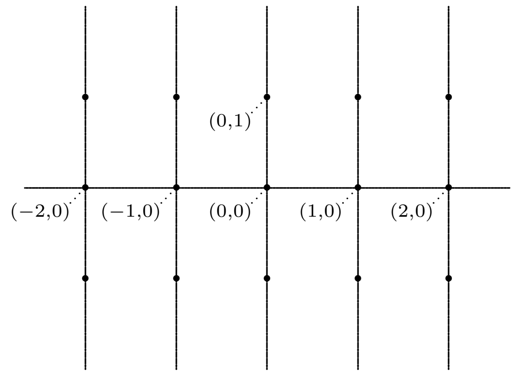
Proof. It is well known that for every measurable space $(X, \mathcal{M}, \mu)$ and for every measurable function g, $|\int_X g(x) \, \mathrm{d}\mu(x)| \le \int_X |g(x)| \, \mathrm{d}\mu(x)$ and it can be easily shown that equality holds if and only if there exists $\theta \in [0, 2\pi)$ such that $|g(x)| = g(x) e^{i\theta}$ μ-almost everywhere. If we apply the first fact to our case, that is to $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, $\mu(\{n\}) = a_n$ and $g(n) = z^n$, then $\int_X g(x) \, \mathrm{d}\mu(x) = \sum_{n=1}^\infty a_n z^n = f(z)$ and the first part of the claim is proved.

Equality obviously holds if $z = |z|$. Now suppose equality holds: this can be only if there exists $\theta \in [0, 2\pi)$ such that $|z|^n = z^n e^{i\theta}$ for all n such that $a_n \ne 0$. By assumption, there exists $k \in \mathbb{N}$ such that either $a_{2k} \ne 0$ and $a_{2k+1} \ne 0$ or $a_{2k-1} \ne 0$ and $a_{2k} \ne 0$. If we have the first case, then $|z|^{2k} = z^{2k} e^{i\theta}$ and $|z|^{2k+1} = z^{2k} e^{i\theta}$. Hence there exist $m, p \in \mathbb{N}$ such that $2k \arg(z) + \theta = 2m\pi$ and $(2k+1)\arg(z) + \theta = 2p\pi$, that is $\arg(z) = 2(m-p)\pi$ and $z = |z|$. The proof in the case $a_{2k-1} \ne 0$, $a_{2k} \ne 0$ is completely analogous. □

3. Local limit theorems and generating functions

The 2-*comb lattice* C_2 is a *spanning tree* of $\mathbb{Z}^2$, that is a subgraph of $\mathbb{Z}^2$ which is a tree and contains all vertices. More precisely, we obtain the 2-comb by deleting in $\mathbb{Z}^2$ all edges that are parallel to the x-axis besides those on the x-axis itself. This construction provides a natural choice of coordinates on the comb (that is $(x,y) \in C_2$ indicates the same point of $\mathbb{Z}^2$, now thought as belonging to C_2). Alternatively, we construct it by attaching at each point of $\mathbb{Z}$ a two way-infinite path (that is a copy of $\mathbb{Z}$).

Figure 1: the 2-comb



More generally, d-dimensional comb lattices C_d are the spanning trees of $\mathbb{Z}^d$ obtained inductively by attaching at each point of C_{d-1} a copy of $\mathbb{Z}$.

The estimate of the asymptotic behaviour of the transition probabilities of the simple random walk on comb lattices passes through the knowledge of the generating functions, which we now recall.

Given two vertices of the comb, x and y, and the Markov chain $(Z_n)_n$ representing the simple random walk on the comb starting at x, the n-step transition probabilities are defined by

$$p^{(n)}(x, y) = \mathbb{P}[Z_n = y | Z_0 = x].$$

The Green function is then the power series

$$G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \quad z \in \mathbb{C},$$

while

$$F(x, y|z) = \sum_{n=0}^{\infty} \mathbb{P}[\mathbf{s}^y = n | Z_0 = x] z^n, \quad z \in \mathbb{C},$$

where $\mathbf{s}^y := \inf[n \geq 0 | Z_n = y]$. Then it is well known (see for instance [2] and [3]) that in the common domain of convergence of these power series $G(x, y|z) = F(x, y|z)G(y, y|z)$.

The Green function of the d-dimensional comb, G_d, can be obtained recursively by the following formula (see [4])

$$G_d(z) = \frac{d}{\sqrt{\left(1 + \frac{d-1}{G_{d-1}(z)}\right)^2 - z^2}}, \tag{3.1}$$

recalling that $G_1(z) = 1/\sqrt{1 - z^2}$.

From equation (3.1) it is easy to see that $G_d(z) = G_d(-z)$, and that $\{+1, -1\}$ are algebraic singularities (see [5] pag. 498) since these properties hold for G_1.

By using mathematical induction on eq. (3.1), it is not difficult to see that

$$G_d(z) \overset{z \to 1^{\pm}}{\sim} d 2^{1/2^d - 1} (1 \pm z)^{-1/2^d}. \tag{3.2}$$

Local limit theorems, that is numerical estimates of $p^{(n)}(o, o)$ where by o we denote the origin of the grid $\mathbb{Z}^d$ where C_d is embedded, were studied by Weiss and Havlin (see [6]) in the case $d = 2$ or $d = 3$, and extended to the general case by Cassi and Regina (see [4]). In fact applying Theorem 4 of [5] to equation (3.2) we have that

$$p^{(n)}(o, o) \overset{n}{\sim} \begin{cases} 0 & \text{if } n \text{ is odd} \\[2ex] \dfrac{2^{1/2^d + 1}}{\Gamma\left(\frac{1}{2^d}\right)} n^{1/2^d - 1} & \text{if } n \text{ is even.} \end{cases}$$

From this particular estimate one can derive an estimate of the general transition probabilities (see Bertacchi and Zucca [1] Paragraph 6):

$$p^{(n)}(x,y) \overset{n}{\sim} \begin{cases} 0 & \text{if } n+d(x,y) \text{ is odd} \\ \dfrac{2^{1/2^d-1}\deg(y)}{\Gamma\left(\frac{1}{2^d}\right)} n^{1/2^d-1} & \text{if } n+d(x,y) \text{ is even.} \end{cases} \tag{3.3}$$

Our goal is to obtain asymptotic estimates of $p^{(n)}((0,k),(0,0))$ and of $p^{(n)}((k,0),(0,0))$, $k \geq 0$, *uniform* with respect to the parameter $\xi = \frac{k}{n}$ (in order to avoid discussions about the parity of n and k we will only deal with the case where they both are even, but this is no severe restriction — see Paragraphs 4 and 10).

We will make use of Cauchy's integral formula

$$p^{(n)}(x,y) = \frac{1}{2\pi i}\int_\gamma \frac{G(x,y|z)}{z^{n+1}}\,\mathrm{d}z$$

(γ is a positively oriented, simple closed curve in $\mathbb{C}$ with 0 in its interior), and then estimate this integral. Hence we need the explicit expressions of $G((k,0),(0,0)|z)$ and $G((0,k),(0,0)|z)$, which are the following:

$$G\big((k,0),(0,0)|z\big) = \frac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}}\cdot\left(\frac{1+\sqrt{1-z^2}-\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}}{z}\right)^{|k|}$$

$$G\big((0,k),(0,0)|z\big) = \frac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}}\cdot\left(\frac{1-\sqrt{1-z^2}}{z}\right)^{|k|}.$$

Computation uses well known techniques for the generating functions on graphs involving explicit expressions of $G(o,o|z)$, $F((k,0),(0,0)|z)$ and $F((0,k),(0,0)|z)$ (see for instance Woess [3] Theorem 1.4).

Now for simplicity we rename

$$\begin{aligned} &\widetilde{G}(z) := \tfrac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}}; && G(z) := \widetilde{G}(\sqrt{z}) = \tfrac{\sqrt{2}}{\sqrt{1-z+\sqrt{1-z}}}; \\ &\widetilde{F}_1(z) := \tfrac{1-\sqrt{1-z^2}}{z}; && F_1(z) := \widetilde{F}_1(\sqrt{z}) = \tfrac{1-\sqrt{1-z}}{\sqrt{z}}; \\ &\widetilde{F}_2(z) := \tfrac{1+\sqrt{1-z^2}-\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}}{z}; && F_2(z) := \widetilde{F}_2(\sqrt{z}) = \tfrac{1+\sqrt{1-z}-\sqrt{2}\sqrt{1-z+\sqrt{1-z}}}{\sqrt{z}}. \end{aligned}$$

We note that these generating functions all contain radicals, hence we must pay attention to their polidromy.
Remark The functions $G(z)$, $F_1^2(z)$ and $F_2^2(z)$ are holomorphic in the open ball with radius 1 centered in 0, that is, 0 is not a singularity for any of these functions. Moreover, $z=1$ is a branch point for all of them and their only singularity in the complex plane.

Choice of the determination of the square root: we choose, once for all, the determination of the square root with argument between $-\pi/2$ and $\pi/2$, that is the function $h(w) := \sqrt{|w|}\exp(i\arg(w)/2)$ where $\arg(w)$ is chosen in the interval $[-\pi,\pi)$. That means that $\sqrt{1-z}$ is an holomorphic function defined in the open set $A := \mathbb{C}\setminus\{z\in\mathbb{R} : z>1\}$, and we extend it to $\{z\in\mathbb{R} : z>1\}$ by continuity from the upper half plane (then we will not have continuity from the lower half plane). Note that this choice allows us to define $\sqrt{1-z+\sqrt{1-z}}$ as an holomorphic function in A as well.

4. Estimates along the y-axis: the case $\xi \in [a, 1-c]$

We first deal with the case of the asymptotic estimate for the transition probabilities $p^{(n)}((0,k),(0,0))$, which we will rename for sake of simplicity $p^{(n)}(k,0)$. We first note that if n and k do not have the same parity, then $p^{(n)}(k,0)=0$, while

$$p^{(2n+1)}(2k+1,0) = \frac{1}{2}\{p^{(2n)}(2k+2,0)+p^{(2n)}(2k,0)\}.$$

Hence it will be enough to estimate $p^{(2n)}(2k,0)$.

By Cauchy's integral formula,

$$p^{(2n)}(2k,0) = \frac{1}{2\pi i}\int_{\gamma'} \frac{\widetilde{G}(z)\cdot \widetilde{F}_1(z)^{2k}}{z^{2n+1}}\,\mathrm{d}z$$

where γ' is a positively oriented, simple closed curve in $\mathbb{C}$ which has 0 in its interior and 1 in its exterior.

The substitution $u = z^2$ appears to be useful and when $\gamma' : z = z(t)$ describes one circuit about 0 then $\gamma : u = u(t)$ describes two times the corresponding circuit, hence

$$p^{(2n)}(2k,0) = \frac{1}{2\pi i}\int_{\gamma} \frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z,$$

where $F(z) = F_1(z)$.

We first stress the exponential part of the integrand, rewriting

$$\begin{aligned}\frac{F(z)^{2k}}{z^n} &= \exp\{2k\log(F(z)) - n\log(z)\} = \\ &= \exp\{n[2\xi\log(1-\sqrt{1-z}) - (\xi+1)\log(z)]\},\end{aligned}$$

where $\xi := \frac{k}{n}$. We now introduce the auxiliary function

$$\Psi_\xi(z) := 2\xi\log(1-\sqrt{1-z}) - (\xi+1)\log(z) \tag{4.1}$$

and rewrite the expression for the transition probabilities:

$$p^{(2n)}(2k,0) = \frac{1}{2\pi i}\int_{\gamma} \frac{G(z)}{z}\exp\{n\Psi_\xi(z)\}\,\mathrm{d}z. \tag{4.2}$$

Lemma 4.1. *The function $\Psi_\xi(z)$ has a unique minimum in $[0,1]$, namely $z_o(\xi) = 1-\xi^2$. Let $\varphi(\xi)$ be this minimum: then $\varphi(\xi) = \log\left((1-\xi)^{\xi-1}(1+\xi)^{-\xi-1}\right)$. The second and the third order derivatives of $\Psi_\xi(z)$ take the following values in the generic point z and in $z_o(\xi)$:*

$$\begin{aligned}\Psi''_\xi(z) &= \tfrac{2(1-z)^{3/2}+\xi(3z-2)}{2z^2(1-z)^{3/2}}; &\quad \Psi''_\xi(z_o(\xi)) &= \tfrac{1}{2\xi^2(1-\xi^2)};\\ \Psi'''_\xi(z) &= \tfrac{\xi(15z^2-20z+8)-8(1-z)^{5/2}}{4z^3(1-z)^{5/2}}; &\quad \Psi'''_\xi(z_o(\xi)) &= \tfrac{3-7\xi^2}{4\xi^4(1-\xi^2)^2}.\end{aligned}$$

Here is the first estimate of our transition probabilities.

Theorem 4.2. *Let a, c be positive numbers such that $a < 1-c$ and let $\xi \in [a, 1-c]$. Then*

$$p^{(2n)}(2k,0) \overset{n}{\sim} \frac{\xi}{\sqrt{\pi}\sqrt{1-\xi^2}}\,G(z_o(\xi))e^{n\varphi(\xi)}\,n^{-1/2} = \sqrt{\frac{2\xi}{(1-\xi^2)(1+\xi)}}\frac{e^{n\varphi(\xi)}}{\sqrt{\pi}}\,n^{-1/2}$$

uniformly with respect to $\xi \in [a, 1-c]$.

Proof. We first choose the curve of integration for equation (4.2) and split the integral into two parts (Part *I* of the proof); then we evaluate the part which will prove to be asymptotically negligible as compared to the other (Part *II* of the proof) and finally estimate the main part (Part *III* of the proof).

Part I

The curve of integration will be the circle with radius $z_o(\xi)$, centered in the origin: $\gamma : z(\xi,t) = z_o(\xi)e^{it},\ t \in [-\pi,\pi]$. We note that since $\xi \in [a, 1-c]$ then $z_o(\xi) \in [1-(1-c)^2, 1-a^2] =: [\overline{a}, 1-\overline{c}]$.

We introduce the function $\overline{\Psi}_\xi(t) := \Psi_\xi(z_o(\xi)e^{it}) = \Psi_\xi(z(\xi,t))$ and perform the change of variable $z = z(\xi,t)$ in the integral (4.2):

$$p^{(2n)}(2k,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(z(\xi,t))\exp\{n\overline{\Psi}_\xi(t)\}\,\mathrm{d}t. \tag{4.3}$$

Now we want to write the Taylor expansion of $\overline{\Psi}_\xi(t)$ with Lagrange remainder, centered in $t=0$. This is possible since the third order derivative of $\overline{\Psi}_\xi(t)$ exists and is continuous in t, for all $\xi \in [a, 1-c]$. Hence we can write $\overline{\Psi}_\xi(t) = \varphi(\xi) - \frac{1-\xi^2}{4\xi^2}t^2 + R(\xi,t)$, where the remainder is $R(\xi,t) = \frac{\overline{\Psi}'''_\xi(\bar{t})}{3!}t^3$, and $\bar{t}$ is a point lying in the segment between 0 and t.

In particular we note that $\overline{\Psi}'''_\xi(0)$ is bounded and far from 0 for $\xi \in [a,1-c]$, that is there exists $\varepsilon > 0$ such that $|\overline{\Psi}'''_\xi(0)| > \varepsilon$ for all $\xi \in [a,1-c]$. Then $R(\xi,t) = O(t^3)$ for $t \to 0$, uniformly with respect to $\xi \in [a,1-c]$, that is, there exists $C > 0$ such that $|R(\xi,t)| \le Ct^3$ for all $\xi \in [a,1-c]$ and t sufficiently small. Moreover, this implies that $R(\xi,t) = o(t^2)$ uniformly with respect to $\xi \in [a,1-c]$, that is for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\left|\frac{R(\xi,t)}{t^2}\right| < \varepsilon$ for all $(\xi,t) \in [a,1-c] \times [-\delta,\delta]$.

Hence we can choose $\alpha > 0$ such that $|R(\xi,t)| \le -\frac{\overline{\Psi}''_\xi(0)}{4}t^2$ (remember that $\overline{\Psi}''_\xi(0)$ exists, is finite and strictly negative for all $\xi \in [a,1-c]$).

Now we split the integral in (4.3) into two parts: the first with t ranging from $-\alpha$ to α and the second will be the rest:

$$\begin{aligned}(\mathbf{A}) &:= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} G(z(\xi,t))\exp\{n\overline{\Psi}_\xi(t)\}\,\mathrm{d}t,\\ (\mathbf{B}) &:= \frac{1}{2\pi}\int_{\alpha<|t|\le\pi} G(z(\xi,t))\exp\{n\overline{\Psi}_\xi(t)\}\,\mathrm{d}t.\end{aligned}$$

Part II

We estimate **(B)**. Let us note that, by the definitions of φ and $z_o(\xi)$,

$$\exp\{n\varphi(\xi)\} = \exp\{n\Psi(z_o(\xi))\} = \frac{F(z_0(\xi))^{2k}}{z_o(\xi)^n}$$

from which we get $z_o(\xi)^n = \frac{F(z_0(\xi))^{2k}}{\exp\{n\varphi(\xi)\}}$. We also note that $\exp\{n\overline{\Psi}_\xi(t)\} = \frac{F(z(\xi,t))^{2k}}{z(\xi,t)^n} = \frac{F(z(\xi,t))^{2k}}{z_o(\xi)^n e^{int}}$. Hence we rewrite **(B)**:

$$(\mathbf{B}) = \frac{e^{n\varphi(\xi)}}{2\pi}\int_{\alpha<|t|\le\pi} G(z_o(\xi)e^{it})\left(\frac{F(z_o(\xi)e^{it})}{F(z_o(\xi))}\right)^{2n\xi} e^{-int}\,\mathrm{d}t.$$

We want to give an upper estimate for the modulus of the integrand: we claim that $|F(z)| \le F(|z|)$ and equality holds if and only if $z = |z|$ (apply Proposition 2.5). Now we observe that $F(z)$ is a continuous mapping and $F(|z|) = 0$ if and only if $z = 0$, then $\frac{|F(z)|}{F(|z|)}$ is continuous in the compact set $K := \{z \in \mathbb{C} : |z| \in [\overline{a}, 1-\overline{c}], \alpha \le \arg(z) \le \pi\}$ where it attains a maximum $\lambda < 1$. Then since $\xi < 1$,

$$|(\mathbf{B})| \le \frac{e^{n\varphi(\xi)}}{2\pi}\int_{\alpha<|t|\le\pi} |G(z_o(\xi)e^{it})|\lambda^{2n}\,\mathrm{d}t.$$

Moreover, by Proposition 2.5, $|G(z_o(\xi)e^{it})| \le G(1-\overline{c})$, from which we get the upper estimate (uniform with respect to $\xi \in [a,1-c]$):

$$|(\mathbf{B})| \le e^{n\varphi(\xi)}G(1-\overline{c})\lambda^{2n}. \tag{4.4}$$

Part III

We estimate **(A)**:

$$(\mathbf{A}) = \frac{e^{n\varphi(\xi)}}{2\pi}\int_{-\alpha}^{\alpha} G(z_o(\xi)e^{it})\exp\left\{-n\frac{1-\xi^2}{4\xi^2}t^2 + nR(\xi,t)\right\}\,\mathrm{d}t.$$

We perform a change of variable in order to stress the main term of the exponential: $\theta := \sqrt{n} b(\xi) t$, where $b(\xi) = \sqrt{\frac{1-\xi^2}{2\xi^2}} = \sqrt{-\overline{\Psi}''_\xi(0)}$. Hence $\mathrm{d}t = \frac{n^{-1/2}\,\mathrm{d}\theta}{b(\xi)}$ and **(A)** becomes

$$(\mathbf{A}) = \frac{e^{n\varphi(\xi)}}{2\pi b(\xi)} G(z_o(\xi)) n^{-1/2} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{\theta^2}{2} + nR(\xi, t_n)\right\} \left(\frac{G(z_o(\xi)e^{it_n})}{G(z_o(\xi))}\right) \mathrm{d}\theta,$$

where we put $t_n := \theta/(\sqrt{n}b(\xi))$. We want to give a uniform upper bound for the modulus of the integrand: we will then be able to apply Theorem 2.3 . As we noted before, $\left|\frac{G(z_o(\xi)e^{it_n})}{G(z_o(\xi))}\right| \leq 1$, moreover, by our choice of α, we have that $|nR(\xi, t_n)| \leq n \frac{-\overline{\Psi}''_\xi(0)}{4} t_n^2 = \frac{\theta^2}{4}$. Then the modulus of the integrand is bounded by $\exp\{-\theta^2/4\}$ for all n, uniformly with respect to $\xi \in [a, 1-c]$; the integrand converges pointwise to $\exp\{-\theta^2/2\}$, and the interval of integration converges to $\mathbb{R}$. Then applying Theorem 2.3 and noting that $\int_{\mathbb{R}} \exp\{-\theta^2/2\} = \sqrt{2\pi}$, we have that

$$(\mathbf{A}) \overset{n}{\sim} \frac{e^{n\varphi(\xi)}}{\sqrt{\pi}} n^{-1/2} \frac{\xi}{\sqrt{1-\xi^2}} G(z_o(\xi)) = \frac{e^{n\varphi(\xi)}}{\sqrt{\pi}} n^{-1/2} \sqrt{\frac{\xi}{1-\xi^2}} \sqrt{\frac{2}{1+\xi}},$$

uniformly with respect to $\xi \in [a, 1-c]$. To conclude the proof, we only need to show the asymptotical negligibility of **(B)**, that is, that $\left|\frac{(\mathbf{B})}{(\mathbf{A})}\right| \to 0$ when n tends to infinity, uniformly with respect to $\xi \in [a, 1-c]$. In fact, by equation (4.4)

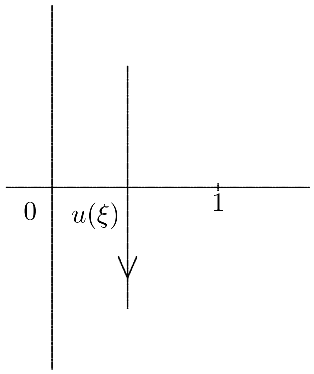
$$\begin{aligned} \left|\frac{(\mathbf{B})}{(\mathbf{A})}\right| &\leq \frac{e^{n\varphi(\xi)}\lambda^{2n} G(1-\overline{c})}{(\mathbf{A})} \overset{n}{\sim} \sqrt{\pi} \frac{\sqrt{1-\xi^2}}{\xi} \lambda^{2n} \frac{G(1-\overline{c})}{G(z_o(\xi))} n^{1/2} \leq \\ &\leq \sqrt{\pi} n^{1/2} \lambda^{2n} \frac{G(1-\overline{c})}{G(\overline{a})} \frac{\sqrt{1-a^2}}{a} \end{aligned}$$

and the last term tends to 0 when n tends to infinity, uniformly with respect to $\xi \in [a, 1-c]$. □

5. Estimates along the y-axis: the case $\xi \in [0, a]$

If ξ is allowed to tend to zero, the preceding estimate is no longer true. Then we have to choose a different curve of integration. We perform a change of variable $u = f(z)$. The curve of integration will be the union of two pieces: one will be the inverse image of a suitable curve in the u-plane, and the other will be a part of a circle centered in the origin such that the union of the two pieces forms a connected circuit about the origin.

We choose $u := \sqrt{1-z}$, which, by our choice of the determination of the square root, has argument in $[-\pi/2, \pi/2)$, and then if $\mathbf{Re}\,(u) < 0$ or $u = ib$, $b > 0$, then $u = -\sqrt{1-z}$. Anyway, $z = 1 - u^2$. The desired curve in the u-plane is simply a vertical segment whose parametrization is $u(\xi, t) = u(\xi) - it$ where $u(\xi) := \sqrt{1 - z_o(\xi)} = \xi$ and t ranges from $-\alpha$ to α (α will be chosen in the sequel). Note that the segment is oriented downwards in order to produce a correctly oriented curve in the z-plane (see the pictures below) and that it lies in the half plane where $u = \sqrt{1-z}$.



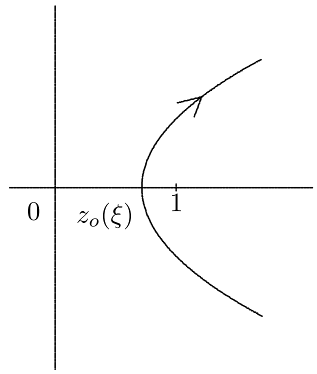


Figure 2: the segment in the u-plane *Figure 3: the curve in the z-plane*

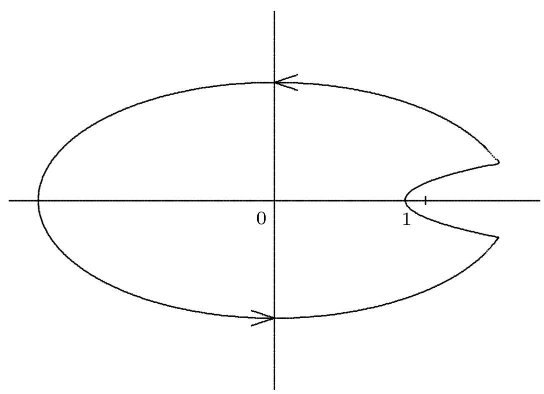
The curve of integration in the z-plane will be the union of $z(\xi,t) := 1 - u(\xi,t)^2$ for $|t| \le \alpha$ and $\tilde{z}(\xi,s) := |z(\xi,\alpha)|e^{is}$ for $\arg(z(\xi,\alpha)) \le s \le 2\pi - \arg(z(\xi,\alpha))$ (note that $\arg(z(\xi,\alpha)) = -\arg(z(\xi,-\alpha))$). Hence

$$p^{(2n)}(2k,0) = (\mathbf{A}) + (\mathbf{B}) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\gamma_2} \frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z \tag{5.1}$$

where γ_1 corresponds to $z(\xi,t)$, $|t| \le \alpha$, and γ_2 corresponds to $\tilde{z}(\xi,s)$, $\arg(z(\xi,\alpha)) \le s \le 2\pi - \arg(z(\xi,\alpha))$ (and **(A)** and **(B)** are the corresponding integrals).

Here is how the contour of integration appears in the z-plane (note that the circumference is elliptic due to a different choice of measure units on the horizontal and vertical axes).

Figure 4: the curve of integration in the z-plane



We observe that the integral still makes sense, since $G(z)$ has a unique singularity in $z = 1$ and can be extended to an holomorphic function defined in an open set containing the integration domain. Similarly, also $F(z)^{2k}$ can be extended to an holomorphic function defined in an open set containing the integration domain (this appears obvious once we look at the explicit expressions for these functions instead of their power series representation).

First we choose a depending on α such that for all $\xi \in [0,a]$ we have $|z(\xi,\alpha)| \ge 1 + \varepsilon_o$ for some fixed $\varepsilon_o > 0$. Thanks to this choice, the circular part of the curve of integration will be uniformly far from the singularity $z = 1$ (that is its distance from 1 is greater than $\varepsilon > 0$, not depending on ξ). This choice is possible since the mapping $\xi \mapsto z(\xi,\alpha)$ is continuous and if $\xi \to 0$ then $z(\xi,\alpha) \to 1 + \alpha^2$. Then for all $\varepsilon > 0$ there exists a right neighbourhood $\mathcal{U}$ of 0 such that whenever $\xi \in \mathcal{U}$ we have $|z(\xi,\alpha)| \ge 1 + \alpha^2 - \varepsilon$ and the last quantity is greater than $1 + \varepsilon_o$ for ε sufficiently small.

We observe that in this paragraph we will operate further choices of a, namely a will be chosen sufficiently small in order to satisfy all the conditions we will find out to be necessary. In the sequel we will not stress that when a new condition is introduced, if necessary a is chosen smaller than that of the preceding choice.

Now we estimate the integral **(B)** that will appear to be asymptotically negligible if compared to the integral **(A)**.

Lemma 5.1. *Let γ_2 be the curve with parametrization $z(t) := |z(\xi,\alpha)|\exp(it)$ where $|t| \ge \arg(z(\xi,\alpha))$.*

Then there exists a sufficiently small a such that

$$\left|\frac{1}{2\pi i}\int_{\gamma_2}\frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z\right| \le C\,e^{n\varphi(\xi)}\,\lambda^n$$

for some $C > 0$, $\lambda < 1$ and for all $\xi \in [0, a]$.

Proof. Let us write the integrand as a function of t:

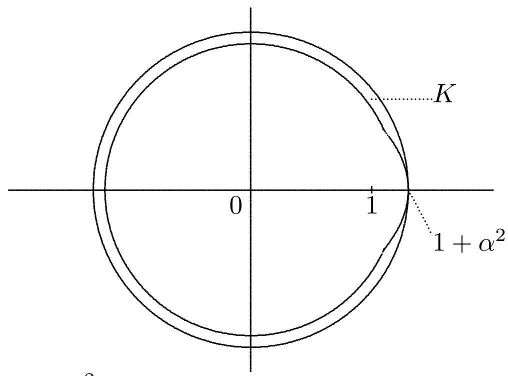
$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma_2}\frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z = \\ &= \frac{e^{n\varphi(\xi)}}{2\pi}\int_{|t|\ge \arg(z(\xi,\alpha))}\frac{(F(|z(\xi,\alpha)|e^{it}))^{2n\xi}}{F(z_o(\xi))^{2n\xi}}\cdot\frac{z_o(\xi)^n}{|z(\xi,\alpha)|^n}\cdot\frac{G(|z(\xi,\alpha)|e^{it})}{e^{itn}}\,\mathrm{d}t\end{aligned} \tag{5.2}$$

where we used $F(z_o(\xi))^{2k} = e^{n\varphi(\xi)}\cdot z_o(\xi)^n$.

We want to give an upper bound for $|G|$ and $|F^2|$, uniformly with respect to $\xi \in [0, a]$ and $|t| \ge \arg(z(\xi, \alpha))$ (this upper bound will depend on a).

This is possible since $\bigcup_{\xi\in[0,a]}\{(\xi, t) : |t| \ge \arg(z(\xi, \alpha))\}$ is a compact subset of $\mathbb{R}^2$ and the mapping $(\xi, t) \mapsto |z(\xi, \alpha)|e^{it}$ is continuous, hence $K := \{z = |z(\xi, \alpha)|e^{it} : \xi \in [0, a], |t| \ge \arg(z(\xi, \alpha))\}$ is a compact subset of $\mathbb{C}$.

Figure 5: the compact set K



Moreover $d(K, 1) \ge \varepsilon_o$ and G and F^2 are holomorphic functions in an open domain containing K, therefore there exists $C \ge 1$ (depending on a) such that

$$\max_{z\in K}(|F^2(z)|, |G(z)|) \le C. \tag{5.3}$$

Hence the modulus of the integrand in equation (5.2) is bounded by

$$\left(\frac{C}{F(z_o(a))}\right)^{na}\cdot\left(\frac{1}{1+\varepsilon_o}\right)^{n}\cdot C.$$

We observe that we may write $C = C(a)$ and if we suppose to choose $C(a)$ to be the smallest possible $C \ge 1$ satisfying (5.3), then $C(a)$ turns out to be a continuous increasing function (since as a increases, $K = K(a)$ becomes a larger subset of $\mathbb{C}$). We want to show that, if a is small enough, then

$$\lambda := \left(\frac{C(a)}{F(z_0(a))}\right)^{a}\cdot\left(\frac{1}{1+\varepsilon_o}\right) < 1. \tag{5.4}$$

We note that $a \mapsto \left(\frac{C(a)}{F(z_0(a))}\right)^a$ is a continuous increasing function, not smaller than 1, and $\lim_{a\to 0} z_o(a) = 1$, F is continuous and $F(1) = 1$, hence

$$\lim_{a\to 0}\left(\frac{C(a)}{F(z_0(a))}\right)^{a} = 1.$$

As $\frac{1}{1+\varepsilon_o} < 1$, a can be chosen small enough in order to satisfy (5.4). The thesis follows using this estimate. $\square$

In order to estimate the integral **(A)**, we rewrite it stressing the exponential part of the integrand:

$$(\mathbf{A}) = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \exp\{n\overline{\Psi}_\xi(t)\}\frac{G(z(\xi,t))}{z(\xi,t)}(u(\xi)-it)\,\mathrm{d}t$$

where $z(\xi,t) = 1-(u(\xi)-it)^2$ and $\overline{\Psi}_\xi(t) := \Psi_\xi(z(\xi,t))$.

Lemma 5.2. *The function $\overline{\Psi}_\xi(t)$ has a Taylor expansion centered in 0:*

$$\overline{\Psi}_\xi(t) = \varphi(\xi) - \frac{1}{2}\left(\frac{2}{1-u(\xi)^2}\right)t^2 + R(\xi,t)$$

where $|R(\xi,t)| \le C|t|^3$ for all $\xi \in [0,a]$, $|t| \le \alpha$, and for some $C > 0$. (C depends on a and α).

Proof. We calculate the first derivative of $\overline{\Psi}_\xi$: its Taylor series will lead us to the Taylor expansion of the primitive function:

$$\overline{\Psi}'_\xi(t) = -i\left\{\frac{1}{1+i\frac{t}{1-u(\xi)}} - \frac{1}{1-i\frac{t}{1+u(\xi)}}\right\} = \sum_{n\ge1}\left[\frac{(-i)^{n+1}}{(1-u(\xi))^n} + \frac{(i)^{n+1}}{(1+u(\xi))^n}\right]t^n,$$

where the series converges to the function itself, provided that $\left|\frac{t}{1\pm u(\xi)}\right| < 1$ (but this is true for a and α sufficiently small). Then

$$\overline{\Psi}_\xi(t) = \sum_{n\ge1}\left[\frac{(-i)^{n+1}}{(1-u(\xi))^n} + \frac{(i)^{n+1}}{(1+u(\xi))^n}\right]\frac{t^{n+1}}{n+1} + \varphi(\xi)$$

where we added the constant $\varphi(\xi) = \overline{\Psi}_\xi(0)$, and

$$\overline{\Psi}_\xi(t) = \varphi(\xi) - \frac{1}{1-u(\xi)^2}t^2 + \frac{4}{3}i\frac{u(\xi)}{(1-u(\xi)^2)^2}t^3 + t^4\,S(\xi,t)$$

where $S(\xi,t) = \sum_{n\ge4}\frac{1}{n}\left[\frac{(-i)^n}{(1-u(\xi))^{n-1}} + \frac{(i)^n}{(1+u(\xi))^{n-1}}\right]t^{n-4}$. We want to show that $|S(\xi,t)| \le C$ where $C > 0$ does not depend on $\xi \in [0,a]$ nor on $t \in [-\alpha,\alpha]$. This will conclude the proof, since

$$R(\xi,t) := \left[\frac{4}{3}i\frac{u(\xi)}{(1-u(\xi)^2)^2} + tS(\xi,t)\right]t^3.$$

But

$$|S(\xi,t)| \le \sum_{n=4}^{\infty}\frac{1}{n}\left[\frac{1}{(1-a)^{n-1}}+1\right]|t|^{n-4} \le \sum_{n=0}^{\infty}\left(\frac{2-a}{1-a}\right)^{n+3}\frac{\alpha^n}{n+4}.$$

Now the last term is equal to a constant $C = C(a,\alpha)$ not depending on ξ, provided that the power series converges in α. Moreover the radius ρ of this power series is $\frac{2-a}{1-a} \le 4$ if $a \le \frac{1}{2}$ and therefore $\rho \ge \frac{1}{4}$. Choosing $\alpha \le \frac{1}{4}$ we get to the conclusion. $\square$

The asymptotic estimate turns out to be different whether ξ is allowed to tend very fast to 0 or not, that is we have to distinguish two subcases.

Theorem 5.3. *If $\xi \in [n^{-1/4}, a]$ for some $a \in (0,1)$, then*

$$p^{(2n)}(2k,0) \overset{n}{\sim} \sqrt{\frac{2}{\pi}}\frac{e^{n\varphi(\xi)}\sqrt{\xi}}{\sqrt{(1+\xi)(1-\xi^2)}}\,n^{-1/2}$$

uniformly with respect to $\xi \in [n^{-1/4}, a]$.

Proof. We rewrite the integral **(A)** using the Taylor expansion of $\overline{\Psi}_\xi(t)$:

$$(\mathbf{A}) = \frac{e^{n\varphi(\xi)}}{\pi} \int_{-\alpha}^{\alpha} \frac{G(z(\xi,t))}{z(\xi,t)} \exp\left\{ n \left[-\frac{1}{2} \left(\frac{2}{1-u(\xi)^2} \right) t^2 + R(\xi,t) \right] \right\} (u(\xi) - it)\, \mathrm{d}t. \tag{5.5}$$

Now we perform a change of variable in order to stress the main term of the exponential: $\theta := \sqrt{n}b(\xi)t$, where $b(\xi) = \sqrt{\frac{2}{1-u(\xi)^2}}$. Moreover we denote by $t_n = \theta/(\sqrt{n}b(\xi))$. Then we choose α (possibly smaller than the preceding choice) such that

$$|R(\xi,t)| \le \frac{1}{4} b(\xi)^2 t^2 \qquad \forall |t| \le \alpha \tag{5.6}$$

uniformly with respect to $\xi \in [0,a]$. This is possible thanks to a similar argument as in the case $\xi > a$ (see Theorem 4.2 Part I), noting that $b(\xi) \ge b(0) = \sqrt{2}$. After the change of variable

$$(\mathbf{A}) = \frac{e^{n\varphi(\xi)}}{\pi\sqrt{n}b(\xi)} \frac{G(z_o(\xi))}{z_o(\xi)} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \frac{G\left(z\left(\xi,t_n\right)\right)}{G(z_o(\xi))} \frac{z_o(\xi)}{z\left(\xi,t_n\right)} \cdot$$
$$\cdot \exp\left\{ -\frac{1}{2}\theta^2 + nR\left(\xi,t_n\right) \right\} \left(u(\xi) - it_n\right) \mathrm{d}\theta.$$

We want to give a uniform (with respect to $\xi \in [n^{-1/4}, a]$) upper bound for the modulus of the integrand in order to apply Theorem 2.3. The exponential part is easily bounded:

$$\left| \exp\left\{ -\frac{1}{2}\theta^2 + nR\left(\xi,t_n\right) \right\} \right| \le \exp\left\{ -\frac{1}{2}\theta^2 + n\frac{b(\xi)^2}{4} t_n^2 \right\} = \exp\{-\theta^2/4\}, \tag{5.7}$$

while $u(\xi) - it_n \overset{n}{\sim} u(\xi) = \xi$ uniformly with respect to $\xi \in [n^{-1/4}, a]$, for every fixed θ, as n tends to infinity.

Let us note that $z_o(\xi) = |z_o(\xi)| \le |z(\xi,t)|$ for all $|t| \le \alpha$, $\xi \in [n^{-1/4}, a]$, hence

$$\left| \frac{z_0(\xi)}{z(\xi,t_n)} \right| \le 1 \tag{5.8}$$

uniformly with respect to $\xi \in [n^{-1/4}, a]$. Now we only have to evaluate $\left| \frac{G(z(\xi,t_n))}{G(z_o(\xi))} \right|$. But, using $u(\xi) = \xi$, we have

$$\left| \frac{G(z(\xi,t_n))}{G(z_o(\xi))} \right| = \frac{\sqrt{\xi(1+\xi)}}{\left|\sqrt{(\xi - it)(1+\xi-it)}\right|} \le 1 \tag{5.9}$$

for all $\xi \in [0,a]$, $|t| \le \alpha$. Then

$$(\mathbf{A}) = \frac{e^{n\varphi(\xi)}u(\xi)}{\pi\sqrt{n}b(\xi)} \frac{G(z_o(\xi))}{z_o(\xi)} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \frac{G(z(\xi,t_n))}{G(z_o(\xi))} \frac{z_o(\xi)}{z(\xi,t_n)} \cdot$$
$$\cdot \exp\left\{ -\frac{1}{2}\theta^2 + nR(\xi,t_n) \right\} \left(1 - i\frac{\theta}{u(\xi)\sqrt{n}b(\xi)} \right) \mathrm{d}\theta$$

where the modulus of the integrand is uniformly bounded with respect to $\xi \in [n^{-1/4}, a]$ by $\exp\{-\theta^2/4\}(1 + c|\theta|)$ (use (5.7), (5.8), (5.9)) and the integrand converges pointwise to $e^{-\frac{1}{2}\theta^2}$. Then we apply Theorem 2.3 and we obtain

$$(\mathbf{A}) \overset{n}{\sim} \frac{e^{n\varphi(\xi)}u(\xi)}{\pi\sqrt{n}b(\xi)} \frac{G(z_o(\xi))}{z_o(\xi)} \int_{\mathbb{R}} e^{-\frac{1}{2}\theta^2}\, \mathrm{d}\theta = \sqrt{\frac{2}{\pi}} \frac{e^{n\varphi(\xi)}\sqrt{\xi}}{\sqrt{(1+\xi)(1-\xi^2)}}\, n^{-1/2}$$

uniformly with respect to $\xi \in [n^{-1/4}, a]$. To prove the thesis, we only have to show asymptotic negligibility of **(B)** with respect to **(A)**. This follows from Lemma 5.1:

$$\left|\frac{(\mathbf{B})}{(\mathbf{A})}\right| \leq \frac{e^{n\varphi(\xi)}\lambda^n C}{\sqrt{\frac{2}{\pi}} \frac{e^{n\varphi(\xi)}\sqrt{\xi}}{\sqrt{(1+\xi)(1-\xi^2)}}}\, n^{-1/2} \leq C\lambda^n n^{1/2+1/8}$$

where the last term tends to zero as n tends to infinity, uniformly with respect to $\xi \in [n^{-1/4}, a]$. □

When $\xi \in [0, n^{-1/4}]$ it is no longer true that $u(\xi) - it_n = \xi - it_n \sim \xi$ and the technique will be slightly different. A useful tool will appear to be the computation of the real and imaginary parts of $\sqrt{\xi - it_n}$.

Lemma 5.4. *Let $\sqrt{\xi - it} = a(\xi, t) + ib(\xi, t)$. Then*

$$a(\xi, t) = \sqrt{\frac{\sqrt{\xi^2 + t^2} + \xi}{2}}, \qquad b(\xi, t) = \operatorname{sign}(-t)\sqrt{\frac{\sqrt{\xi^2 + t^2} - \xi}{2}}$$

and both these terms are $O(\sqrt{\xi}) + O(\sqrt{|t|})$.

Theorem 5.5. *If $\xi \in [0, n^{-1/4}]$ then*

$$p^{(2n)}(2k, 0) \overset{n}{\sim} \frac{e^{n\varphi(\xi)}}{\sqrt{2\pi}} I\left(\sqrt{n}\xi\right)\, n^{-3/4},$$

where $I(t) := \int_{\mathbb{R}} e^{-\frac{\theta^2}{2}} \sqrt{\sqrt{t^2 + \frac{\theta^2}{2}} + t}\, \mathrm{d}\theta$, and the estimate is uniform with respect to $\xi \in [0, n^{-1/4}]$. Moreover, if $\xi \in [0, n^{-1/2-\varepsilon}]$ for some $\varepsilon > 0$, then

$$p^{(2n)}(2k, 0) \overset{n}{\sim} \frac{\sqrt{2}\, e^{n\varphi(\xi)}}{\Gamma\left(\frac{1}{4}\right)}\, n^{-3/4},$$

uniformly with respect to $\xi \in [0, n^{-1/2-\varepsilon}]$.

Proof. The integral we have to estimate is still **(A)** of equation (5.5), α is chosen as in Theorem 5.3 and we perform the same change of variable $\theta := \sqrt{n}b(\xi)t$. Then we proceed differently: in Part *I* we show a decomposition of $G(z)$ into its singular and regular parts; Part *II* is devoted to the estimate of the real part of the integrand in **(A)**: we stress only the terms which are not $o(\xi)$ or $o(t_n^2)$ (uniformly with respect to $\xi \in [0, n^{-1/4}]$). In Part *III* we write $(\mathbf{A}) = (\mathbf{A_1}) + (\mathbf{A_2}) + (\mathbf{A_3})$. In Part *IV* we estimate $(\mathbf{A_1})$ and describe some properties of $I(t)$. Finally, in Part *V* we show negligibility of **(B)**, $(\mathbf{A_2})$ and $(\mathbf{A_3})$.

Part I

We write a decomposition for $G(z)$:

$$\begin{aligned} G(z) &= \frac{\sqrt{2}}{(1-z)^{1/4}}(1 + \sqrt{1-z})^{-1/2} = \frac{\sqrt{2}}{(1-z)^{1/4}} \cdot \sum_{n=0}^{\infty} \binom{-1/2}{n} (1-z)^{n/2} = \\ &= (1-z)^{-1/4} H(z) + (1-z)^{1/4} K(z) \end{aligned} \tag{5.10}$$

where $H(z) := \sqrt{2}\sum_{n=0}^{\infty} \binom{-1/2}{2n} (1-z)^n$ and $K(z) := \sqrt{2}\sum_{n=0}^{\infty} \binom{-1/2}{2n+1} (1-z)^n$ are two holomorphic functions defined in a disc centered in $z = 1$ and with greater or equal to 2/3 radius. Note that the expansion of $(1 + \sqrt{1-z})^{-1/2}$ as a power series is admissible provided that $|1-z| < 1$, and this condition surely holds in the integration domain if α is sufficiently small.

We decompose H and K into their real and imaginary parts: $H = H_0 + iH_1$ and $K = K_0 + iK_1$. The following properties hold: $H(1) = H_0(1) = \sqrt{2}$, $K(1) = K_0(1) = -1/\sqrt{2}$, $H_1(z) = O(|1-z|)$, $K_1(z) = O(|1-z|)$ and $O(|1-z|) = O(\xi^2) + O(t_n^2)$.

Part II

We can write **(A)** as follows:

$$(\mathbf{A}) = \frac{e^{n\varphi(\xi)}}{\pi\sqrt{n}b(\xi)} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR(\xi,t_n)\right\} \frac{\sqrt{\xi - it_n}}{z(\xi,t_n)} \cdot$$
$$\cdot \left(H(z(\xi,t_n)) + (\xi - it_n)K(z(\xi,t_n))\right) \,\mathrm{d}\theta.$$

Since the transition probabilities are non negative quantities, we are interested only in the real part of the last integral (in fact the imaginary part must be 0, but taking into account only the real part of the integrand avoids useless computation).

Our aim is to apply Theorem 2.4, then we want to estimate the function (depending on ξ, n and θ) to which the real part of our integrand is asymptotic (uniformly with respect to $\xi \in [0, n^{-1/4}]$). Hence we rewrite the integrand using $\sqrt{\xi - it_n} = a(\xi,t_n) + ib(\xi,t_n)$ (see Lemma 5.4):

$$\mathbb{1}_{[-\alpha\sqrt{n}b(\xi),\alpha\sqrt{n}b(\xi)]}(\theta) \cdot \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi,t_n)\right\} \exp\left\{inR_1(\xi,t_n)\right\} \cdot$$
$$\cdot \frac{1}{z_n}(a(\xi,t_n) + ib(\xi,t_n)) \left(H(z_n) + (a(\xi,t_n) + ib(\xi,t_n))^2 K(z_n)\right),$$

where $R_0(\xi,t_n)$ and $R_1(\xi,t_n)$ are respectively the real and imaginary part of $R(\xi,t_n)$ and $z_n := z(\xi,t_n)$.

First we want to estimate the main term of

$$\mathbf{Re}\ \left\{\exp\left\{inR_1(\xi,t_n)\right\}(a(\xi,t_n) + ib(\xi,t_n)) \left(H(z_n) + (a(\xi,t_n) + ib(\xi,t_n))^2 K(z_n)\right)\right\}. \tag{5.11}$$

It is useful to note that

$$nR_1(\xi,t_n) = t_n(O(\xi) + O(t_n)) \tag{5.12}$$

for every fixed θ, uniformly with respect to $\xi \in [0, n^{-1/4}]$. In fact,

$$R_1(\xi,t) = \left[\frac{4}{3}\frac{u(\xi)}{(1-u(\xi))^2} + t\mathbf{Im}\ S(\xi,t)\right] t^3 = [O(\xi) + O(t)]t^3 \tag{5.13}$$

since

$$S(\xi,t) = \frac{1}{4}\left[(1-u(\xi))^{-3} + (1+u(\xi)^{-3}\right] + t\sum_{n\geq 0}\frac{1}{n+5}\left[\frac{(-i)^{n+1}}{(1-u(\xi))^{n+4}} + \frac{(i)^{n+1}}{(1+u(\xi))^{n+4}}\right] t^n$$

and hence $\mathbf{Im}\ S(\xi,t) = O(t)$ uniformly with respect to $\xi \in [0, n^{-1/4}]$.

Then from (5.13) we get

$$nR_1(\xi,t_n) = nt_n^2 \cdot t_n(O(\xi) + O(t_n)) = \frac{\theta^2}{b(\xi)^2} t_n(O(\xi) + O(t_n))$$

which leads to (5.12) since $b(\xi) \geq \sqrt{2}$ for all $\xi \in [0,a]$.

Now we expand the product in (5.11), where we distinguish the real and imaginary parts of H and K and write a_n and b_n for $a(\xi,t_n)$ and $b(\xi,t_n)$ respectively. We claim that this product can be written as follows:

$$\cos(nR_1(\xi,t_n))a_nH_0(z_n) + \cos(nR_1(\xi,t_n))b_nt_nK_0(z_n) + o(\xi) + o(t_n^2), \tag{5.14}$$

where $o(\xi)$ and $o(t_n^2)$ are uniform with respect to $\xi \in [0, n^{-1/4}]$.

The proof of (5.14) is tedious but straightforward; particular care should be put only in estimating the following terms:

$$\begin{aligned} \xi \cdot O(\sqrt{|t_n|}) &= o(\xi), \\ O(\sqrt{\xi}) \cdot O(t_n^2) &= o(t_n^2), \\ O(\xi) \cdot O(|t_n|^{3/2}) &= o(\xi), \end{aligned} \tag{5.15}$$

and these estimates are uniform with respect to $\xi \in [0, n^{-1/4}]$, since we consider n tending to infinity, which implies that both ξ and t_n tend to 0 (for every fixed θ).

Finally, we note that $z_n^{-1} = 1 + O(\xi^2) + O(t_n^2)$ for every fixed θ, uniformly with respect to $\xi \in [0, n^{-1/4}]$:

$$\left| \frac{1}{z_n} - 1 \right| = \frac{|\xi - it_n|^2}{|1 - (\xi - it_n)^2|} \le C\,(\xi^2 + t_n^2). \tag{5.16}$$

Then by (5.14) and (5.16) we can write **(A)** as follows:

$$\begin{aligned}(\mathbf{A}) = {} & \frac{e^{n\varphi(\xi)}}{\pi\sqrt{n}b(\xi)} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi, t_n)\right\} (1 + O(\xi^2) + O(t_n^2)) \cdot \\ & \cdot \left\{\cos(nR_1(\xi, t_n)) a_n H_0(z_n) + \cos(nR_1(\xi, t_n)) b_n t_n K_0(z_n) + o(\xi) + o(t_n^2)\right\} \,\mathrm{d}\theta,\end{aligned}$$

where all the "O" and "o" are uniform with respect to $\xi \in [0, n^{-1/4}]$, for every fixed θ.

Part III

We divide **(A)** in three parts, and we write a_n and b_n as functions of θ, ξ and n:

$$\begin{aligned} a_n &= \frac{(1-\xi^2)^{1/4}}{\sqrt{2}\, n^{1/4}} \sqrt{\sqrt{\frac{\theta^2}{2} + \frac{n\xi^2}{(1-\xi^2)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}}} \\ b_n &= -\frac{\theta(1-\xi^2)^{1/4}}{2n^{1/4}\sqrt{\sqrt{\frac{\theta^2}{2} + \frac{n\xi^2}{(1-\xi^2)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}}}}, \end{aligned}$$

obtaining

$$\begin{aligned} (\mathbf{A_1}) = {} & \frac{e^{n\varphi(\xi)}(1-\xi^2)^{3/4}}{\pi\sqrt{2}\, n^{3/4}} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi, t_n)\right\} (1 + O(\xi^2) + O(t_n^2)) \cdot \\ & \cdot \cos(nR_1(\xi, t_n)) \frac{1}{\sqrt{2}} \sqrt{\sqrt{\frac{\theta^2}{2} + \frac{n\xi^2}{(1-\xi^2)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}}} H_0(z_n)\,\mathrm{d}\theta, \\ (\mathbf{A_2}) = {} & \frac{e^{n\varphi(\xi)}(1-\xi^2)^{3/4}}{\pi\sqrt{2}\, n^{3/4}} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi, t_n)\right\} (1 + O(\xi^2) + O(t_n^2)) \cdot \\ & \cdot \cos(nR_1(\xi, t_n)) \frac{\theta}{2\sqrt{\sqrt{\frac{\theta^2}{2} + \frac{n\xi^2}{(1-\xi^2)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}}}} \frac{\theta\sqrt{1-\xi^2}}{\sqrt{2n}} (-K_0(z_n))\,\mathrm{d}\theta \\ (\mathbf{A_3}) = {} & \frac{e^{n\varphi(\xi)}(1-\xi^2)^{1/2}}{\pi\sqrt{2}\, n^{1/2}} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi, t_n)\right\} (1 + O(\xi^2) + O(t_n^2)) \cdot \\ & \cdot (o(\xi) + o(t_n^2))\,\mathrm{d}\theta. \end{aligned} \tag{5.17}$$

Part IV

We notice that the integral in $(\mathbf{A_1})$ seems to be asymptotic to

$$I\left(\sqrt{n}\xi\right) = \int_{\mathbb{R}} e^{-\frac{\theta^2}{2}} \sqrt{\sqrt{n\xi^2 + \frac{\theta^2}{2}} + \sqrt{n}\xi}\,\mathrm{d}\theta,$$

then it will be useful to state some properties of $I(t)$: $I(t)$ exists and is finite for every $t \ge 0$, it is continuous and increasing in $[0, +\infty)$, differentiable in $(0, +\infty)$, $I(t) \ge 2\sqrt{t\pi}$ and if t tends to infinity, $I(t) \sim 2\sqrt{t\pi}$.

Now we want to apply Theorem 2.4 to the integrand of $(\mathbf{A_1})$ in (5.17), but a new problem raises: the quantity $\sqrt{n}\xi$ may tend to 0 or to $+\infty$ as well. Hence we introduce a new function, namely $Q(n,\xi) := \max\{1, \sqrt{n}\xi\} \geq 1$ ($Q(n,\xi)$ tends to infinity if and only if $\sqrt{n}\xi \to +\infty$). Once again we rewrite $(\mathbf{A_1})$:

$$\frac{e^{n\varphi(\xi)}(1-\xi^2)^{3/4}\sqrt{Q(n,\xi)}}{\pi\sqrt{2}n^{3/4}} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi,t_n)\right\}\cdot$$
$$\cdot\,(1+O(\xi^2)+O(t_n^2))cos(nR_1(\xi,t_n))\frac{1}{\sqrt{2}}H_0(z_n)\cdot$$
$$\cdot\,\sqrt{\sqrt{\frac{\theta^2}{2Q^2(n,\xi)} + \frac{n\xi^2}{(1-\xi^2)Q^2(n,\xi)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}Q(n,\xi)}}\,\mathrm{d}\theta.$$

We must first evaluate the (uniform) asymptotic value of the integrand: recalling that $|nR_0(\xi,t_n)| \to 0$ for every fixed θ, uniformly with respect to $\xi \in [0, n^{-1/4}]$, it is easy to show that

$$f_n(\theta,\xi) := \mathbb{1}_{[-\alpha\sqrt{n}b(\xi),\alpha\sqrt{n}b(\xi)]}(\theta)\exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi,t_n)\right\}\cdot$$
$$\cdot\,(1+O(\xi^2)+O(t_n^2))cos(nR_1(\xi,t_n))\frac{1}{\sqrt{2}}H_0(z_n)\cdot$$
$$\cdot\,\sqrt{\sqrt{\frac{\theta^2}{2Q^2(n,\xi)} + \frac{n\xi^2}{(1-\xi^2)Q^2(n,\xi)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}Q(n,\xi)}} \sim$$
$$\sim\; h_n(\theta,\xi) := \exp(-\theta^2/2)\cdot$$
$$\cdot\,\sqrt{\sqrt{\frac{\theta^2}{2Q^2(n,\xi)} + \frac{n\xi^2}{(1-\xi^2)Q^2(n,\xi)}} + \frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}Q(n,\xi)}}.$$

As for a uniform bound for f_n and h_n, they are both uniformly bounded by

$$C\,\exp(-\theta^2/4)\sqrt{\sqrt{1+\theta^2/2}+1}.$$

Finally, $|\int_{\mathbb{R}} h_n(\theta,\xi)\,\mathrm{d}\theta| > c$ for some $c > 0$, for all n and $\xi \in [0, n^{-1/4}]$, since $h_n(\theta,\xi) \geq e^{-\theta^2/2}\cdot\min\{c, 2^{-1/4}\sqrt{|\theta|}\}$.

Hence, by Theorem 2.4,

$$(\mathbf{A_1}) \overset{n}{\sim} = \frac{e^{n\varphi(\xi)}}{\pi\sqrt{2}n^{3/4}} I\left(\sqrt{n}\xi\right).$$

Part V

We are now able to show that **(B)** is asymptotically negligible if compared to $(\mathbf{A_1})$, using Lemma 5.1:

$$\left|\frac{(\mathbf{B})}{(\mathbf{A_1})}\right| \leq \frac{Ce^{n\varphi(\xi)}\lambda^n}{e^{n\varphi(\xi)}n^{-3/4}I(0)} \leq C\lambda^n n^{3/4}$$

and the last term tends to 0 uniformly with respect to $\xi \in [0, n^{-1/4}]$.

We estimate $(\mathbf{A_2})$ and $(\mathbf{A_3})$ to show their negligibility too:

$$|(\mathbf{A_2})| = \frac{e^{n\varphi(\xi)}(1-\xi^2)^{5/4}}{4\pi n^{5/4}} \int_{-\alpha\sqrt{n}b(\xi)}^{\alpha\sqrt{n}b(\xi)} \exp\left\{-\frac{1}{2}\theta^2 + nR_0(\xi,t_n)\right\}(1+O(\xi^2)+O(t_n^2))\cdot$$
$$\cdot\,\cos(nR_1(\xi,t_n))\frac{\theta^2}{\sqrt{\sqrt{\frac{\theta^2}{2}+\frac{n\xi^2}{(1-\xi^2)}}+\frac{\sqrt{n}\xi}{\sqrt{1-\xi^2}}}}(-K_0(z_n))\,\mathrm{d}\theta$$
$$\leq C\frac{e^{n\varphi(\xi)}}{n^{5/4}}\int_{\mathbb{R}}\exp(-\theta^2/4)\frac{\theta^2}{|\theta|^{1/2}}\,\mathrm{d}\theta$$

since $\sqrt{\sqrt{t^2+\theta^2/2}+t} \geq 2^{-1/4}|\theta|^{1/2}$ for all $t \geq 0$. Then

$$\left|\frac{(\mathbf{A_2})}{(\mathbf{A_1})}\right| \leq C\frac{e^{n\varphi(\xi)}n^{-5/4}}{e^{n\varphi(\xi)}n^{-3/4}} \leq Cn^{-1/2}$$

which tends uniformly to 0 with respect to $\xi \in [0, n^{-1/4}]$.

As for $(\mathbf{A_3})$, we observe that every $o(\xi)$ besides those in (5.15) is also equal to $\xi^{1/8}o(\xi)$, while for the $o(\xi)$ in (5.15) we have that

$$\begin{aligned}
\frac{\xi \cdot O(\sqrt{|t_n|})}{\xi} &\leq C\sqrt{|t_n|} \leq C\frac{\theta^{1/2}}{n^{1/4}}, \\
\frac{O(\xi)\cdot O(|t_n|^{3/2})}{\xi} &\leq C|t_n|^{3/2} \leq C\frac{\theta^{3/2}}{n^{3/4}}.
\end{aligned}$$

Moreover, $|O(\xi^2)|$ and $|O(t_n^2)|$ are smaller than some constant $C>0$ (recall that $|t_n| \leq \alpha$ and $\xi \leq n^{-1/4}$), and $|o(t_n^2)| \leq C\,\frac{\theta}{n}$, hence we can majorize $|(\mathbf{A_3})|$:

$$|(\mathbf{A_3})| \leq C\, e^{n\varphi(\xi)}n^{-3/4}\left(\xi^{1/8}+n^{-1/4}\right)$$

whence

$$\left|\frac{(\mathbf{A_3})}{(\mathbf{A_1})}\right| \leq C\,\frac{e^{n\varphi(\xi)}n^{-3/4}\left(\xi^{1/8}+n^{-1/4}\right)}{e^{n\varphi(\xi)}n^{-3/4}I(0)} \leq C\,\left(\xi^{1/8}+n^{-1/4}\right)$$

and the last term tends uniformly to 0 with respect to $\xi \in [0, n^{-1/4}]$.

The statement for $\xi \in [0, n^{-1/2-\varepsilon}]$ is proved in the same way, once we note that

$$I(0) = \sqrt{2}\Gamma\left(\frac{3}{4}\right), \text{ and } \Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)}$$

(use $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and duplication formula $\Gamma(2z)=(2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$). $\square$

6. Estimates along the x-axis: the case $\xi \in [a, 1-c]$

In this section we want to give an asymptotic estimate for the transition probabilities $p^{(n)}((k,0),(0,0))$, which we will rename $p^{(n)}(k,0)$. We first note that if n and k do not have the same parity, then $p^{(n)}(k,0)=0$, but we cannot repeat the trick we used on the y-axis to derive $p^{(2n+1)}(2k+1,0)$ from $p^{(2n)}(2k,0)$. We estimate only the second type of transition probabilities (the first ones can be derived in a similar way — see Paragraph 10).

The basic idea underlying our proofs here is essentially the same as for the case of the y-axis, nevertheless much more technical difficulties will arise and the techniques we will employ appear more involved. We want to point out that here we will use the same symbols (such as F, Ψ_ξ, φ) for functions and values which are not the same but play the same role as the analogous for the y-axis.

By Cauchy's integral formula,

$$p^{(2n)}(2k,0) = \frac{1}{2\pi i}\int_{\gamma'}\frac{\widetilde{G}(z)\cdot\widetilde{F}_2(z)^{2k}}{z^{2n+1}}\,\mathrm{d}z$$

where γ' is a positively oriented, simple closed curve in $\mathbb{C}$ which has 0 in its interior and 1 in its exterior.

The substitution $u=z^2$ appears to be useful and when $\gamma' : z=z(t)$ describes one circuit about 0 then $\gamma : u=u(t)$ describes two times the corresponding circuit, then

$$p^{(2n)}(2k,0) = \frac{1}{2\pi i}\int_{\gamma}\frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z$$

where $F(z) = F_2(z)$.

We first stress the "exponential part" of the integrand, rewriting

$$\frac{F(z)^{2k}}{z^n} = \exp\{n[2\xi \log(1 + \sqrt{1-z} - \sqrt{2}\sqrt{1 - z + \sqrt{1-z}}) - (\xi + 1)\log(z)]\},$$

where $\xi := \frac{k}{n}$. We now introduce the auxiliary function

$$\Psi_\xi(z) := 2\xi \log(1 + \sqrt{1-z} - \sqrt{2}\sqrt{1 - z + \sqrt{1-z}}) - (\xi + 1)\log(z),$$

and rewrite the expression for the transition probabilities:

$$p^{(2n)}(2k, 0) = \frac{1}{2\pi i}\int_\gamma \frac{G(z)}{z}\exp\{n\Psi_\xi(z)\}\,\mathrm{d}z. \tag{6.1}$$

Lemma 6.1. *Let $\varphi(\xi) := \min\{\Psi_\xi(z) : 0 \le z \le 1\}$ and let $z_o(\xi)$ be the (unique) point where this minimum is attained. Then $z_o(\xi) = 1 - u_o(\xi)^2$, where $u_o(\xi) = \xi^{2/3}[54 + 6\sqrt{81 - 6\xi^2}]^{1/3}/6 + \xi^{4/3}[54 + 6\sqrt{81 - 6\xi^2}]^{-1/3}$. The second and the third order derivatives of $\Psi_\xi(z)$ take the following values in $z_o(\xi)$:*

$$\begin{aligned}
\Psi''_\xi(z_o(\xi)) &= \frac{1 + 2z_o(\xi) - \sqrt{1 - z_o(\xi)}}{4z_o(\xi)^2(1 - z_o(\xi))} = \\
&= \frac{2u_o(\xi) + 3}{4u_o(\xi)^2(1 - u_o(\xi))(1 + u_o(\xi))^2}; \\
\Psi'''_\xi(z_o(\xi)) &= \frac{-16u_o(\xi)^2 - 24u_o(\xi) + 19}{16u_o(\xi)^4(1 - u_o(\xi))^2(1 + u_o(\xi))};
\end{aligned}$$

(the expression for the third order derivative appears much more complex as a function of $z_o(\xi)$).

The following theorem is the analogous of Theorem 4.2 for the x-axis (and the proof is just the same).

Theorem 6.2. *Let a, c be positive numbers such that $a < 1 - c$ and let $\xi \in [a, 1-c]$. Then*

$$p^{(2n)}(2k, 0) \overset{n}{\sim} \sqrt{\frac{2}{\pi}}\,\frac{e^{n\varphi(\xi)}\sqrt{1 - z_o(\xi)}\,G(z_o(\xi))}{\sqrt{1 + 2z_o(\xi) - \sqrt{1 - z_o(\xi)}}}\,n^{-1/2}$$

uniformly with respect to $\xi \in [a, 1-c]$.

7. Estimates along the x-axis: the case $\xi \in [0, a]$

In order to choose the curve of integration, we perform a change of variable which is the inverse mapping of $z = f(v)$. The curve of integration will be the union of two pieces: one will be the image of a suitable curve in the v-plane, and the other will be a part of a circle centered in the origin such that the union of the two pieces forms a connected circuit about the origin, that is, as in the case of the y-axis,

$$p^{(2n)}(2k, 0) = (\mathbf{A}) + (\mathbf{B}) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\gamma_2} \frac{G(z)F(z)^{2k}}{z^{n+1}}\,\mathrm{d}z, \tag{7.1}$$

where γ_1 corresponds to a particular curve $z(\xi, t)$, $|t| \le \alpha$, and γ_2 corresponds to $|z(\xi, \alpha)|e^{is}$, $s \in [\arg(z(\xi, \alpha)), 2\pi - \arg(z(\xi, \alpha))]$.

We choose $z = 1 - v^4$, hence $v := \sqrt[4]{1 - z}$ is the inverse mapping if, by our choice of the determination of the root, v has argument in $[-\pi/4, \pi/4)$.

The desired curve in the v-plane is a segment whose parametrization is

$$v(\xi,t)=\begin{cases} v(\xi)+e^{i\beta}t & \text{if } t\in[0,\alpha] \\ v(\xi)-e^{-i\beta}t & \text{if } t\in[-\alpha,0), \end{cases}$$

where $v(\xi):=\sqrt{u_o(\xi)}$, and α and β will be chosen in the sequel. The corresponding curve in the z-plane is $z(\xi,t):=1-v(\xi,t)^4$.

In order to obtain the Taylor series of the function $t\mapsto\overline{\Psi}_\xi(t):=\Psi_\xi(z(\xi,t))$ we introduce the auxiliary function

$$\widetilde{\Psi}_\xi(v):=\Psi_\xi(1-v^4)=2\xi\log(1+\sqrt{v^4}-\sqrt{2}\sqrt{v^4+\sqrt{v^4}})-(\xi+1)\log(1-v^4).$$

Particular attention must be paid in computing the roots ($\sqrt{v^4}$ is not necessarily equal to v^2 nor $\sqrt{a\cdot b}$ is always equal to $\sqrt{a}\cdot\sqrt{b}$). For our purposes we only need the explicit expressions of $\widetilde{\Psi}_\xi(v)$ for $\arg(v)\in[-\pi/4,0]$ (where $\sqrt{v^4}=v^2$, $\sqrt{v^4+v^2}=v\sqrt{1+v^2}$) and for $\arg(v)\in[\pi/4,\pi/2)$ (where $\sqrt{v^4}=-v^2$, $\sqrt{v^4-v^2}=-iv\sqrt{1-v^2}$).

Hence

$$\begin{aligned}\widetilde{\Psi}_{\xi,1}(v)&:=2\xi\log(1+v^2-\sqrt{2}\,v\sqrt{1+v^2})-(\xi+1)\log(1-v^4),\\ \widetilde{\Psi}_{\xi,2}(v)&:=2\xi\log(1-v^2+\sqrt{2}\,iv\sqrt{1-v^2})-(\xi+1)\log(1-v^4),\end{aligned}$$

where $\widetilde{\Psi}_{\xi,1}(v)=\widetilde{\Psi}_\xi(v)$ if $\arg(v)\in[-\pi/4,0]$ and $\widetilde{\Psi}_{\xi,2}(v)=\widetilde{\Psi}_\xi(v)$ if $\arg(v)\in[\pi/4,\pi/2)$.

Lemma 7.1. *The functions $\widetilde{\Psi}_{\xi,1}(v)$ and $\widetilde{\Psi}_{\xi,2}(v)$ have a Taylor series expansion centered in 0, with convergence radius equal to 1:*

$$\widetilde{\Psi}_{\xi,1}(v)=\sum_{n=1}^{\infty}d_n(\xi)\,v^n,\qquad\qquad \widetilde{\Psi}_{\xi,2}(v)=\sum_{n=1}^{\infty}d'_n(\xi)\,v^n$$

where $d_{4n}(\xi)=d'_{4n}(\xi)=1/n$ for every $n\geq 1$; $d_{2n+1}(\xi)=-2\sqrt{2}\,\xi\frac{b_n}{2n+1}$, $d'_{2n+1}(\xi)=2\sqrt{2}\,i\xi\,\frac{(-1)^n b_n}{2n+1}$ for every $n\geq 0$, $b_n:=\sum_{i=0}^{[n/2]}\binom{1/2}{n-2i}$ and all other coefficients are equal to zero.

Lemma 7.2. *The function $\overline{\Psi}_\xi(t)=\widetilde{\Psi}_\xi(v(\xi,t))$ has a Taylor series expansion centered in 0, with positive radius of convergence ρ not depending on ξ:*

$$\overline{\Psi}_\xi(t)=\sum_{m=0}^{\infty}e^{im\beta}g_m(\xi)\,t^m, \tag{7.2}$$

where

$$g_m(\xi)=\sum_{h=m}^{\infty}\binom{h}{m}d_h(\xi)v(\xi)^{h-m}$$

if $\arg(v(\xi,t))\in[-\pi/4,0]$ *and*

$$g_m(\xi)=\sum_{h=m}^{\infty}\binom{h}{m}d'_h(\xi)v(\xi)^{h-m}$$

if $\arg(v(\xi,t))\in[\pi/4,\pi/2]$. *(Note that the series defining $g_m(\xi)$ both converge for every $\xi\in[0,a]$, provided that a is sufficiently small).*

To avoid misunderstandings, we define $\overline{\Psi}_{\xi,j}(t)$, $j=1,2$, as follows:

$$\overline{\Psi}_\xi(t)=\begin{cases}\overline{\Psi}_{\xi,1}(t) & \text{if } \arg(v(\xi,t))\in[-\pi/4,0]\\ \overline{\Psi}_{\xi,2}(t) & \text{if } \arg(v(\xi,t))\in[\pi/4,\pi/2].\end{cases}$$

Lemma 7.3. *The following equalities hold:*

$$\overline{\Psi}_{\xi,1}(t) = \varphi(\xi) + e^{2i\beta}a_2(\xi)t^2 + e^{3i\beta}a_3(\xi)t^3 + e^{4i\beta}a_4(\xi)t^4 + R(\xi,t)$$
$$\overline{\Psi}_{\xi,2}(t) = \varphi(\xi) + e^{2i\beta}a_2'(\xi)t^2 + e^{3i\beta}a_3'(\xi)t^3 + e^{4i\beta}a_4'(\xi)t^4 + R(\xi,t)$$

where the remainder term (which is not necessarily equal in the two expressions) is $R(\xi,t) = O(t^5)$*, uniformly with respect to* $\xi \in [0,a]$*,* $|t| \le \alpha$*. Moreover, if* $\xi \to 0$

$$\begin{array}{ll} a_2(\xi) \sim 3\cdot 2^{2/3}\xi^{2/3}; & a_2'(\xi) \sim 3\cdot 2^{2/3}\xi^{2/3} - i2^{1/3}\xi^{4/3}; \\ a_3(\xi) \sim 2^{11/6}\xi^{1/3}; & a_3'(\xi) \sim 2^{11/6}\xi^{1/3} - i\sqrt{2}/3\,\xi; \\ a_4(\xi) \sim 1; & a_4'(\xi) \sim 1 + i\cdot 7\cdot 2^{-3/2}\xi^{4/3}. \end{array}$$

Now we have to choose the curve of integration (that is, β) in order to obtain that:

(a) $\overline{\Psi}_\xi(t) := \widetilde{\Psi}_\xi(v(\xi,t))$ has a Taylor expansion in a neighbourhood of $t=0$;

(b) the curve $z(\xi,t) = 1 - v(\xi,t)^4$ has some "good properties" (for instance there exists $\alpha > 0$ such that $|z(\xi,\alpha)| \ge 1 + \varepsilon_0$ for some $\varepsilon_0 > 0$ and for every ξ in the considered range);

(c) we can bound $\exp\{n\overline{\Psi}_\xi(t)\}$ with an integrable function.

We observe that, not considering the integral **(B)** in equation (7.1), we have to integrate on the curve $z_1(\xi,t) := 1 - (v(\xi) + e^{i\beta}t)^4$ for $t \in [0,\alpha]$ and on $z_2(\xi,t) := 1 - (v(\xi) + e^{i\beta}t)^4$ for $t \in [-\alpha, 0)$, but $\overline{z_1(\xi,t)} = z_2(\xi,-t)$, that is the second curve is simply the conjugate of the first one. Then we denote by $z(\xi,t)$ the curve $z_1(\xi,t)$ and the sum of the two corresponding integrals will be

$$(\mathbf{A}) = \frac{1}{\pi}\mathbf{Im}\int_0^\alpha \frac{G(z(\xi,t))}{z(\xi,t)}\exp\{n\overline{\Psi}_\xi(t)\}[-4e^{i\beta}(v(\xi) + e^{i\beta}t)^3]\,\mathrm{d}t,$$

since $G(\overline{z}) = \overline{G(z)}$ and $\exp\{n\overline{\Psi}_\xi(-t)\} = \overline{\exp\{n\overline{\Psi}_\xi(t)\}}$ (remember that $\exp\{n\overline{\Psi}_\xi(t)\} = F(z(\xi,t))^{2k}/z(\xi,t)^n$, and $F(\overline{z})^2 = \overline{F(z)^2}$).

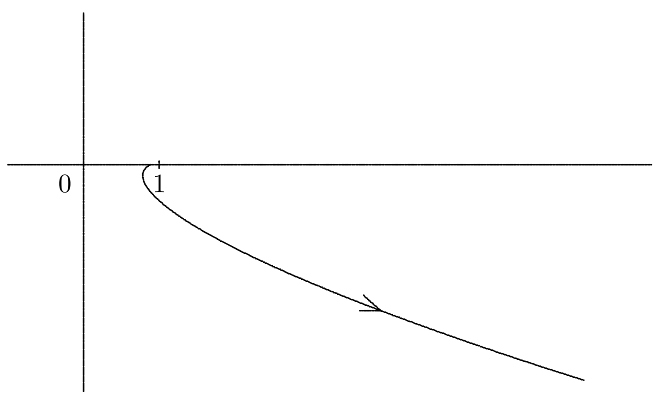
This allows us to consider only the case $t \ge 0$ (that is, the part of the curve that lies in the first quadrant).

As we would like to give an upper bound for the exponential part in **(A)**, we restrict the range of β. In fact we require that the real parts of the expansion in Lemma 7.3 of at least $\overline{\Psi}_{\xi,1}(t)$ have non positive coefficients, that is, that $\cos(2\beta)$, $\cos(3\beta)$ and $\cos(4\beta)$ are all non positive. It is easy to show that this corresponds to $\beta \in \Delta := [\pi/4, 3\pi/8] \cup [-3\pi/8, -\pi/4]$, that is, $e^{i\beta}$ lies in two conjugate sectors.

Remark Among the curves of the family $\mathcal{F} := \{z(\xi,t) \,:\, t \ge 0,\ \beta \in \Delta\}$, the ones with $\beta = \pm\pi/4$ have exactly one intersection with the real axis: $z(\xi,0) = 1 - v(\xi)^4$; while the others have exactly two intersections with the same axis: one for $t=0$ and the other for $t = \frac{v(\xi)}{\sin\beta - \cos\beta}$ if $\beta \in (\pi/4, 3\pi/8]$ or for $t = -\frac{v(\xi)}{\sin\beta + \cos\beta}$ if $\beta \in [-3\pi/8, -\pi/4]$.

We present the plot of some of the curves we will use.

Figure 6: the curve for $\beta = \frac{\pi}{4}$



The plot shows that the curve for $\beta = \frac{\pi}{4}$ turns clockwise (with respect to the origin), then we will prefer its conjugate, that is the curve for $\beta = -\frac{\pi}{4}$.

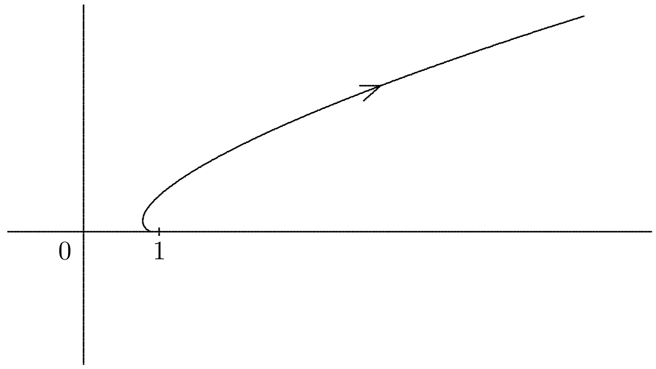


Figure 7: the curve for $\beta = -\frac{\pi}{4}$

Here is the curve for $\beta = \frac{\pi}{3}$, where we used a logarithmic scale on the horizontal axis in order to show the "pathological" behaviour of the curves of the family with $\beta \neq \pm\pi/4$.

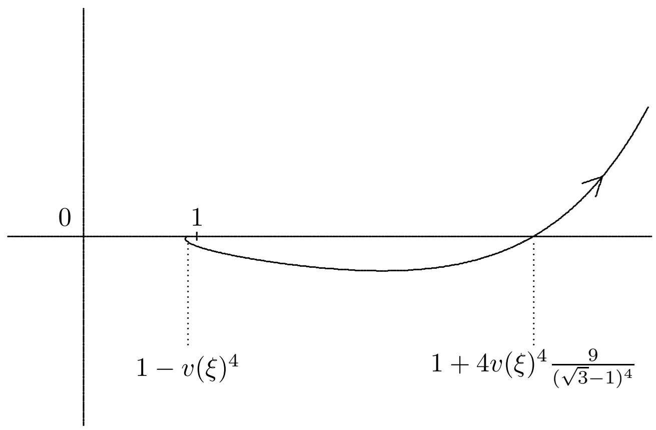


Figure 8: the curve for $\beta = \frac{\pi}{3}$

Now we have to choose a proper change of variable to perform in the integral **(A)**. The choice is between three substitutions, each stressing a different piece of the exponential part of the integrand:
(a) $na_2(\xi)t^2 = \theta^2$
(b) $na_3(\xi)t^3 = \theta^3$
(c) $na_4(\xi)t^4 = \theta^4$

Lemma 7.4. *If we want to have an upper bound for $n\overline{\Psi}_\xi(t(\theta))$, then substitution (a) will do in the case $\xi \geq n^{-3/4}$, substitution (c) in the case $\xi \leq n^{-3/4}$.*

Proof. With substitution (a) and $t = t_n = t_n(\theta) := \frac{\theta}{\sqrt{na_2(\xi)}}$, $\exp\{n\overline{\Psi}_\xi(t)\}$ becomes either

$$e^{n\varphi(\xi)} \exp\left\{e^{2i\beta}\theta^2 + e^{3i\beta}\frac{a_3(\xi)}{\sqrt{n}\,(a_2(\xi))^{3/2}}\,\theta^3 + e^{4i\beta}\frac{a_4(\xi)}{n(a_2(\xi))^2}\,\theta^4 + nR(\xi, t_n)\right\}$$

or

$$e^{n\varphi(\xi)} \exp\left\{e^{2i\beta}\frac{a_2'(\xi)}{a_2(\xi)}\,\theta^2 + e^{3i\beta}\frac{a_3'(\xi)}{\sqrt{n}\,(a_2(\xi))^{3/2}}\,\theta^3 + e^{4i\beta}\frac{a_4'(\xi)}{n(a_2(\xi))^2}\,\theta^4 + nR(\xi, t_n)\right\}$$

(depending on where the point $v(\xi, t)$ lies in the complex plane). We choose a sufficiently small that for every $\xi \in [0, a]$, $a_i \in [c \cdot h_i(\xi), C \cdot h_i(\xi)]$ $i = 2, 3, 4$, where $h_i(\xi)$ is the asymptotic value of $a_i(\xi)$ as ξ tends to zero, $0 < c < C$, and the same holds for the modulus of the real and imaginary parts of $a_i'(\xi)$.

Then

$$\frac{a_3(\xi)}{\sqrt{n}\,(a_2(\xi))^{3/2}} \leq \frac{C}{n^{1/2}\xi^{2/3}}; \qquad \frac{a_4(\xi)}{n(a_2(\xi))^2} \leq \frac{C}{n\xi^{4/3}};$$
$$\left|\frac{\mathbf{Re}\ a_3'(\xi)}{\sqrt{n}\,(a_2(\xi))^{3/2}}\right| \leq \frac{C}{n^{1/2}\xi^{2/3}}; \quad \left|\frac{\mathbf{Re}\ a_4'(\xi)}{n(a_2(\xi))^2}\right| \leq \frac{C}{n\xi^{4/3}};$$

which are all surely bounded if $\xi \geq n^{-3/4}$, while

$$\left|\frac{\mathbf{Im}\ a_3'(\xi)}{\sqrt{n}\,(a_2(\xi))^{3/2}}\right| \leq \frac{C}{n^{1/2}}; \quad \left|\frac{\mathbf{Im}\ a_4'(\xi)}{n(a_2(\xi))^2}\right| \leq \frac{C}{n}$$

both tend to zero uniformly with respect to ξ.

With substitution (b), $t = t_n := \frac{\theta}{\sqrt[3]{n a_3(\xi)}}$, and the coefficient of θ^2 is bounded if $\xi \leq n^{-3/4}$, while the coefficient of θ^4 is bounded if $\xi \geq n^{-3/4}$ (the coefficient of θ^3 is $e^{3i\beta}$). This makes substitution (b) not a suitable one.

Finally, with substitution (c), $t = t_n := \frac{\theta}{\sqrt[4]{n a_4(\xi)}}$, and the coefficients of θ^2 and θ^3 are bounded for $\xi \leq n^{-3/4}$, while the coefficient of θ^4 is $e^{4i\beta}$. □

8. Estimates along the x-axis: the case $\xi \in [0, n^{-3/4}]$

We choose $\beta = -\frac{\pi}{4}$: the curve of integration will be similar to that in Figure 8, even if the parabolic piece is here substituted by the union of the arcs of the curves in Figures 12 and 13 corresponding to $t \in [0, \alpha]$.

We choose a (eventually smaller than the preceding choices) depending on α such that for all $\xi \leq a$ we have $|z(\xi, \alpha)| \geq 1 + \varepsilon_o$ for some fixed $\varepsilon_o > 0$ (where $z(\xi, t) = 1 - (v(\xi) + e^{-\pi/4} t)^4$). Thanks to this choice, the circular part of the curve of integration will be far from the singularity $z = 1$. This choice is possible since the mapping $\xi \mapsto z(\xi, \alpha)$ is continuous and if $\xi \to 0$ then $z(\xi, \alpha) \to 1 + \alpha^4$. Then for all $\varepsilon > 0$ there exists a right neighbourhood $\mathcal{U}$ of 0 such that whenever $\xi \in \mathcal{U}$ we have $|z(\xi, \alpha)| \geq 1 + \alpha^2 - \varepsilon$ and the last quantity is greater than $1 + \varepsilon_o$ for ε sufficiently small (for instance $\varepsilon_o = \alpha^4/2$).

Negligibility of **(B)** is proved in the same way as in Lemma 5.1.

Lemma 8.1. *There exists a sufficiently small a such that*

$$\left| \frac{1}{2\pi i} \int_{|s| \geq \arg(z(\xi,\alpha))} \frac{G(|z(\xi,\alpha)| e^{is}) \, (F(|z(\xi,\alpha)| e^{is})^2)^k}{|z(\xi,\alpha)|^n e^{isn}} \, \mathrm{d}s \right| \leq C \, e^{n\varphi(\xi)} \, \lambda^n$$

for some $C > 0$, $\lambda < 1$ and for all $\xi \in [0, a]$.

Theorem 8.2. *If $\xi \in [0, n^{-3/4}]$ then*

$$p^{(2n)}(2k, 0) \overset{n}{\sim} \frac{4 \, e^{n\varphi(\xi)}}{\pi} I(n^{1/4} \xi^{1/3}) \, n^{-3/4},$$

where

$$I(t) := \int_0^{+\infty} \exp(-2^{4/3} t \theta^3 - \theta^4) \cdot [\cos(3 \cdot 2^{2/3} t^2 \theta^2 - 2^{4/3} t \theta^3)(2^{-1/3} t^2 + \theta^2 + 2^{4/3} t \theta) + \\ + \sin(3 \cdot 2^{2/3} t^2 \theta^2 - 2^{4/3} t \theta^3)(\theta^2 - 2^{-1/3} t^2)] \, \mathrm{d}\theta,$$

and the estimate is uniform with respect to $\xi \in [0, n^{-3/4}]$. Moreover, if $\xi \in [0, n^{-3/4-\varepsilon}]$ for some $\varepsilon > 0$, then

$$p^{(2n)}(2k, 0) \overset{n}{\sim} \frac{\sqrt{2} \, e^{n\varphi(\xi)}}{\Gamma\left(\frac{1}{4}\right)} \, n^{-3/4},$$

uniformly with respect to $\xi \in [0, n^{-3/4-\varepsilon}]$.

Proof. We perform the change of variable (c):

$$
\begin{aligned}
(\mathbf{A}) = \frac{-4\, e^{n\varphi(\xi)}}{\pi \sqrt[4]{n a_4(\xi)}} \mathbf{Im} \Bigg\{ e^{-i\pi/4} \int_0^{n^{1/4} a_4(\xi)^{1/4} \alpha} & \frac{G(z(\xi,t_n))}{z(\xi,t_n)} \cdot \\
\cdot \exp\left\{ -i\, \frac{n^{1/2} a_2(\xi)}{a_4(\xi)^{1/2}}\, \theta^2 - \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \frac{n^{1/4} a_3(\xi)}{a_4(\xi)^{3/4}}\, \theta^3 - \theta^4 + nR(\xi,t_n) \right\} & \cdot \\
\cdot\, (v(\xi) + e^{-i\pi/4} t_n)^3 \, \mathrm{d}\theta \Bigg\} = & \\
\frac{-2\sqrt{2}\, e^{n\varphi(\xi)}}{\pi \sqrt[4]{n a_4(\xi)}} \Big(\mathbf{Im}\ - \mathbf{Re} \Big) \int_0^{n^{1/4} a_4(\xi)^{1/4} \alpha} & \frac{G(z(\xi,t_n))}{z(\xi,t_n)} \cdot \\
\cdot \exp\left\{ -i\, \frac{n^{1/2} a_2(\xi)}{a_4(\xi)^{1/2}}\, \theta^2 - \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \frac{n^{1/4} a_3(\xi)}{a_4(\xi)^{3/4}}\, \theta^3 - \theta^4 + nR(\xi,t_n) \right\} & \cdot \\
\cdot\, (v(\xi) + e^{-i\pi/4} t_n)^3 \, \mathrm{d}\theta. &
\end{aligned}
\tag{8.1}
$$

We note that $\arg(v(\xi) + e^{-i\pi/4} t) \in [-\pi/4, 0]$ for all $t \geq 0$, hence we will only need expression $\overline{\Psi}_{\xi,1}(t)$ for $\overline{\Psi}_{\xi}(t)$.

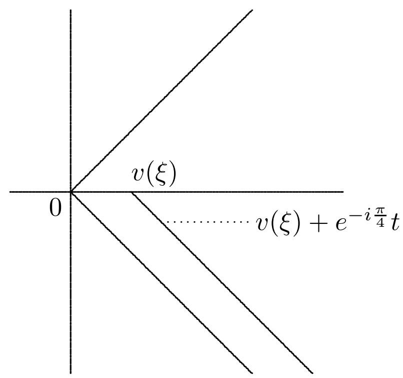


Figure 9: the segment in the v-plane

Now we choose α (smaller than $\frac{1}{\sqrt{6}}$) such that $|R(\xi,t)| \leq \frac{a_4(\xi) t^4}{2}$ for all $\xi \in [0,a]$ and $t \in [0,\alpha]$. This choice is possible since $|R(\xi,t)| \leq C|t|^5$ and $a_4(\xi) > c$ for all $\xi \leq a$. Then $|nR(\xi,t_n)| \leq \frac{\theta^4}{2}$ and we are able to guarantee integrability, since the modulus of the integrand in equation (8.1) is bounded by

$$
C\theta \exp\left\{ -\frac{n^{1/4} a_3(\xi)}{\sqrt{2}\, a_4(\xi)^{3/4}}\, \theta^3 - \frac{\theta^4}{2} \right\} \leq C\theta \exp\left\{ -C'\theta^3 - \frac{\theta^4}{2} \right\}
$$

if $\xi \in [0, n^{-3/4}]$ (we used that $|v(\xi) + e^{-i\pi/4} t_n| \leq C\theta$).

We proceed as in Theorem 5.5: we recall the decomposition of $G(z)$

$$
\begin{aligned}
G(z) &= (1-z)^{-1/4} H(z) + (1-z)^{1/4} K(z) = \\
&= (v(\xi) + e^{-i\pi/4} t)^{-1} H(z(\xi,t)) + (v(\xi) + e^{-i\pi/4} t) K(z(\xi,t))
\end{aligned}
$$

where we used $z = z(\xi,t) = 1 - (v(\xi) + e^{-i\pi/4} t)^4$. Hence the integral in (8.1) can be written as:

$$
\begin{aligned}
\int_0^{n^{1/4} a_4(\xi)^{1/4} \alpha} & \exp\left\{ -\frac{n^{1/4} a_3(\xi)}{\sqrt{2}\, a_4(\xi)^{3/4}}\, \theta^3 - \theta^4 + nR_0(\xi,t_n) \right\} \cdot \\
& \cdot \exp\left\{ -i\, \frac{n^{1/2} a_2(\xi)}{a_4(\xi)^{1/2}}\, \theta^2 - i\, \frac{n^{1/4} a_3(\xi)}{\sqrt{2}\, a_4(\xi)^{3/4}}\, \theta^3 + inR_1(\xi,t_n) \right\} \cdot \\
& \cdot \frac{1}{1 - (v(\xi) + e^{-i\pi/4} t_n)^4} \cdot \\
& \cdot \left\{ (v(\xi) + e^{-i\pi/4} t_n)^2 H(z_n) + (v(\xi) + e^{-i\pi/4} t_n)^4 K(z_n) \right\} \mathrm{d}\theta
\end{aligned}
\tag{8.2}
$$

where $z_n := z(\xi, t_n)$ and R_0 and R_1 are respectively the real and imaginary part of R.

We note that both $H_1(z)$ and $K_1(z)$ are $O(|1-z|)$, that is $O(|v(\xi)+e^{-i\pi/4}t|^4) = o(v(\xi)^2)+o(t^2)$. Moreover, it is easy to show that

$$\frac{1}{1-(v(\xi)+e^{-i\pi/4}t_n)^4} = 1+O(v(\xi)^4)+O(t_n^4) = 1+o(v(\xi)^2)+o(t_n^2),$$

and $o(v(\xi)^2)$ and $o(t^2)$ are uniform with respect to $\xi \in [0, n^{-3/4}]$.

Then we want to pick the main terms in the following product:

$$\begin{aligned}
&\exp\left\{-i\,\frac{n^{1/2}a_2(\xi)}{a_4(\xi)^{1/2}}\,\theta^2 - i\frac{n^{1/4}a_3(\xi)}{\sqrt{2}\,a_4(\xi)^{3/4}}\,\theta^3 + inR_1(\xi,t_n)\right\}\cdot \\
&\quad\cdot(v(\xi)+e^{-i\pi/4}t_n)^2\cdot\left\{H(z_n)+(v(\xi)+e^{-i\pi/4}t_n)^2K(z_n)\right\}.
\end{aligned} \tag{8.3}$$

For simplicity's sake we call $iA_n = iA(n,\xi,\theta)$ the function in the exponential.

Explicit computation shows that the product in (8.3) is:

$$\begin{aligned}
&\Big\{\cos(A_n)\cdot(v(\xi)^2+\sqrt{2}\,v(\xi)t_n)\cdot H_0(z_n)+ \\
&+\sin(A_n)\cdot(\sqrt{2}\,v(\xi)t_n+t_n^2)\cdot H_0(z_n)+o(v(\xi)^2)+o(t_n^2)\Big\}+ \\
&+i\Big\{-\cos(A_n)\cdot(\sqrt{2}\,v(\xi)t_n+t_n^2)\cdot H_0(z_n)+ \\
&+\sin(A_n)\cdot(v(\xi)^2\sqrt{2}+v(\xi)t_n)\cdot H_0(z_n)+o(v(\xi)^2)+o(t_n^2)\Big\}.
\end{aligned}$$

The integral **(A)** becomes:

$$\begin{aligned}
&\frac{2\sqrt{2}\,e^{n\varphi(\xi)}}{\pi\sqrt[4]{a_4(\xi)}\,n^{3/4}}\int_0^{n^{1/4}a_4(\xi)^{1/4}\alpha}\exp\left\{-\frac{n^{1/4}a_3(\xi)}{\sqrt{2}\,a_4(\xi)^{3/4}}\,\theta^3-\theta^4+nR_0(\xi,t_n)\right\}\cdot \\
&\qquad\cdot(1+o(v(\xi)^2)+o(t_n^2))\cdot \\
&\qquad\cdot\Big\{\cos(A_n)\cdot(\sqrt{n}\,v(\xi)^2+2\sqrt{2}\sqrt{n}\,v(\xi)t_n+\sqrt{n}\,t_n^2)\cdot H_0(z_n)+ \\
&\qquad+\sin(A_n)\cdot(\sqrt{n}\,t_n^2-\sqrt{n}\,v(\xi)^2)\cdot H_0(z_n)+\sqrt{n}\,o(v(\xi)^2)+\sqrt{n}\,o(t_n^2)\Big\}\,\mathrm{d}\theta,
\end{aligned} \tag{8.4}$$

where $o(v(\xi)^2)$ and $o(t_n^2)$ are uniform with respect to $\xi\in[0,n^{-3/4}]$ (observe that we divided it by $n^{1/2}$ and multiplied the integrand by the same quantity).

Our aim is to apply Theorem 2.4, then if $f_n(\theta,\xi)$ is the integrand in equation (8.4), we have to evaluate its uniform asymptotic $h_n(\theta,\xi)$, show that there exist $g(\theta)$ and $g_1(\theta)$ in $\mathrm{L}^1(\mathbb{R}^+)$ such that

$$|h_n(\theta,\xi)|\le g(\theta) \text{ and } \qquad |f_n(\theta,\xi)|\le g_1(\theta),$$

for every θ and $\xi\le n^{-3/4}$ and that there exists $c>0$ such that

$$\left|\int_{\mathbb{R}^+} h_n(\theta,\xi)\,\mathrm{d}\theta\right|\ge c \tag{8.5}$$

for all n sufficiently large and $\xi\le n^{-3/4}$. Then we will get that

$$\int_{\mathbb{R}^+} f_n(\theta,\xi)\,\mathrm{d}\theta \;\sim\; \int_{\mathbb{R}^+} h_n(\theta,\xi)\,\mathrm{d}\theta$$

uniformly with respect to $\xi \in [0, n^{-3/4}]$.

Now recalling the definitions of t_n, A_n and z_n, the asymptotic values of $a_i(\xi)$ and $v(\xi)$, and that $H(z_n)$ tends to $\sqrt{2}$ as n tends to infinity,

$$\begin{aligned} f_n(\theta,\xi) \sim h_n(\theta,\xi) =& \sqrt{2}\exp\left\{-2^{4/3}n^{1/4}\xi^{1/3}\,\theta^3 - \theta^4\right\}\cdot \\ & \cdot\left\{\cos\left(3\cdot 2^{2/3}n^{1/2}\xi^{2/3}\theta^2 - 2^{4/3}n^{1/4}\xi^{1/3}\,\theta^3\right)\cdot\right. \\ & \cdot\left(2^{-1/3}n^{1/2}\xi^{2/3} + 2^{4/3}n^{1/4}\xi^{1/3}\theta + \theta^2\right) + \\ & \left. + \sin\left(3\cdot 2^{2/3}n^{1/2}\xi^{2/3}\theta^2 - 2^{4/3}n^{1/4}\xi^{1/3}\,\theta^3\right)\cdot(\theta^2 - 2^{-1/3}n^{1/2}\xi^{2/3})\right\}. \end{aligned}$$

From these formulas it is easy to see that we can use

$$g_1(\theta) = C\exp\left\{-C\theta^3 - \frac{\theta^4}{2}\right\}(1+\theta+\theta^2);$$
$$g(\theta) = C\exp\{-C\theta^3 - \theta^4\}(1+\theta+\theta^2).$$

Now we put $\delta := n^{1/4}\xi^{1/3}$: clearly $\delta \in [0,1]$ and assumption (8.5) is equivalent to $I(\delta) > 0$ for every $\delta \in [0,1]$. But by numerical computation performed with *Derive for Windows* this appears obvious.

Hence by Theorem 2.4

$$(\mathbf{A}) \overset{n}{\sim} \frac{4e^{n\varphi(\xi)}}{\pi} n^{-3/4} I(n^{1/4}\xi^{1/3}),$$

and this is the uniform asymptotic estimate for the $p^{(2n)}(2k,0)$ since by Lemma 8.1

$$\left|\frac{(\mathbf{B})}{(\mathbf{A})}\right| \le C\frac{e^{n\varphi(\xi)}\lambda^n}{e^{n\varphi(\xi)}n^{3/4}\min_{\delta\in[0,1]} I(\delta)} \le C\lambda^n n^{-3/4}$$

which tends to zero uniformly with respect to $\xi \in [0, n^{-3/4}]$. Finally we observe that the same argument conducted this far shows that, if $\xi \in [0, n^{-3/4-\varepsilon}]$ for some $\varepsilon > 0$, the uniform asymptotic estimate is

$$p^{(2n)}(2k,0) \overset{n}{\sim} \frac{4e^{n\varphi(\xi)}}{\pi} n^{-3/4} I(0),$$

and the thesis follows from $I(0) = \int_0^{+\infty} \theta^2 e^{-\theta^4}\,\mathrm{d}\theta = \frac{\pi\sqrt{2}}{4\Gamma(\frac{1}{4})}$. We observe that this last result can be obtained under the weaker assumption that $n^{1/4}\xi^{1/3} \to 0$ uniformly with respect to ξ. □

9. Estimates along the x-axis: the case $\xi \in [n^{-\frac{3}{4}}, a]$

We observe that since in this case we have to use substitution (a), the curve of integration we used so far does not fit. In fact in the exponential, θ^2 would have a pure imaginary coefficient. Hence we are forced to seek another solution, that will be choosing a curve of integration made of more pieces: the curve $z_1(\xi,t) = 1 - (v(\xi) + e^{-i\frac{\pi}{3}}t)^4$ for $0 \le t \le t_o$, $z_2(\xi,t) = 1 - (v(\xi) + e^{i\frac{\pi}{3}}t)^4$ for $t_o \le t \le \alpha$, the conjugate of these two curves and a circular arc to make the whole connected. Here is how these curves appear.

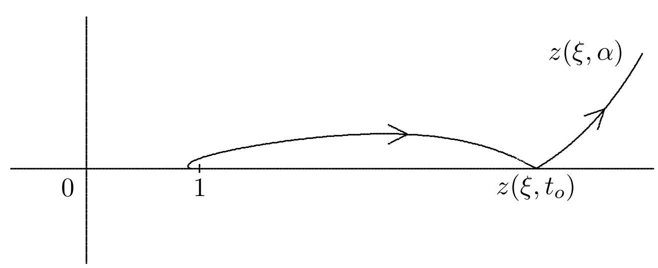


Figure 10: the curve for $t \le \alpha$

And here is a sketch for the curve of integration: we substituted the circumference with an ellipsis (not centered in the origin), only for aesthetical reasons.

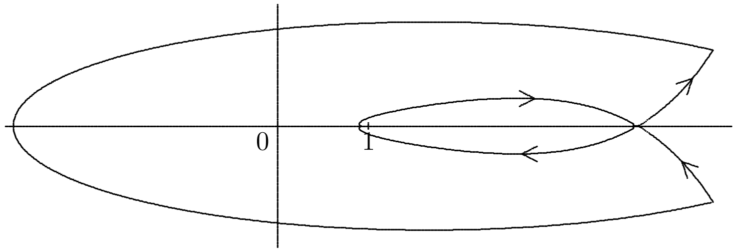


Figure 11: The curve of integration in the case $\xi \geq n^{-\frac{3}{4}}$

We let $z(\xi, t)$ be the expression for the curve of integration for $-\alpha \leq t \leq \alpha$, that is

$$z(\xi,t) = \begin{cases} 1 - (v(\xi) + e^{-i\frac{\pi}{3}}t)^4 & \text{if } 0 \leq t \leq t_o, \\ 1 - (v(\xi) + e^{i\frac{\pi}{3}}t)^4 & \text{if } t_o < t \leq \alpha, \\ 1 - (v(\xi) - e^{i\frac{\pi}{3}}t)^4 & \text{if } -t_o \leq t \leq 0, \\ 1 - (v(\xi) - e^{-i\frac{\pi}{3}}t)^4 & \text{if } -\alpha \leq t < -t_o. \end{cases}$$

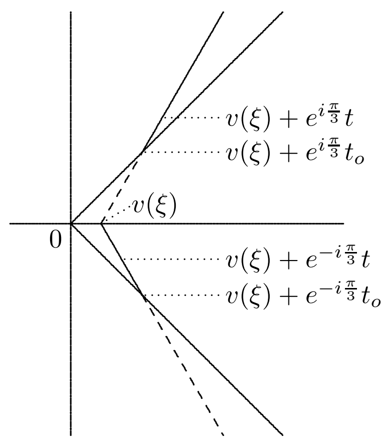
The transition probabilities are given now by equation (7.1) where γ_1 is this new $z(\xi, t)$ and γ_2 is the corresponding circular arc. As we already remarked, integral **(A)** is equal to

$$\frac{1}{\pi}\mathbf{Im} \int_0^\alpha \frac{G(z(\xi,t))}{z(\xi,t)} \exp\{n\overline{\Psi}_\xi(t)\}[-4\, e^{i\beta(t)}(v(\xi) + e^{i\beta(t)}t)^3]\, \mathrm{d}t, \tag{9.1}$$

where $\beta(t) = -\pi/3$ if $t \in [0, t_0]$, $\beta(t) = \pi/3$ if $t \in (t_0, \alpha]$ and $t_0 = 2/(\sqrt{3} - 1)v(\xi)$.

Now the explicit expression for $\overline{\Psi}_\xi(t)$ depends on where in the complex plane the curves $v(\xi, t) = v(\xi) + e^{\pm i\frac{\pi}{3}}t$ lie.

Figure 12: the segments in the v-plane



From the picture above we can see that if $t \in [0, t_o]$ then $v(\xi) + e^{i\frac{\pi}{3}}t$ has argument in the interval $[-\pi/4, 0]$, while if $t \geq t_o$ then $v(\xi) + e^{-i\frac{\pi}{3}}t$ has argument in the interval $[\pi/4, \pi/2)$. Hence in equation (9.1) we can substitute $\overline{\Psi}_\xi(t)$ with the expansion of $\overline{\Psi}_{\xi,1}(t)$ if $t \in [0, t_o]$ and with the expansion of $\overline{\Psi}_{\xi,2}(t)$ otherwise (see Lemma 7.3). Then our goal is the estimate of the imaginary part of

$$\begin{aligned} -4 \int_0^\alpha & \frac{G(z_1(\xi,t))}{z_1(\xi,t)} e^{-i\frac{\pi}{3}}\, (v(\xi) + e^{-i\frac{\pi}{3}}t)^3 \mathbb{1}_{[0,t_o]}(t) \cdot \\ & \cdot \exp\left\{n\varphi(\xi) + ne^{-\frac{2}{3}\pi i}a_2(\xi)t^2 - na_3(\xi)t^3 + ne^{-\frac{4}{3}\pi i}a_4(\xi)t^4 + nR(\xi,t)\right\} + \\ & + \frac{G(z_2(\xi,t))}{z_2(\xi,t)} e^{i\frac{\pi}{3}}\, (v(\xi) + e^{i\frac{\pi}{3}}t)^3 \mathbb{1}_{[t_o,\alpha]}(t) \cdot \\ & \cdot \exp\left\{n\varphi(\xi) + ne^{\frac{2}{3}\pi i}a_2'(\xi)t^2 - na_3'(\xi)t^3 + ne^{\frac{4}{3}\pi i}a_4'(\xi)t^4 + nR'(\xi,t)\right\}\, \mathrm{d}t, \end{aligned}$$

where we wrote $R'(\xi,t)$ for the second remainder term to distinguish it from the first one.

We perform the change of variable $\theta = \sqrt{na_2(\xi)}\,t$ and write t_n for $\frac{\theta}{\sqrt{na_2(\xi)}}$.

We choose α such that $|R'(\xi,t)| \le \frac{\mathbf{Re}\,(a_4'(\xi))}{4}\,t^4$ and $|R(\xi,t)| \le \frac{a_4(\xi)}{4}\,t^4$ for all $\xi \le a$ and $|t| \le \alpha$. Moreover we require that $\alpha > \max_\xi t_o = \frac{2}{\sqrt{3}-1} v(a)$, which is surely true if a is sufficiently small.

We distinguish two subcases: $\xi \in [n^{-3/4}, n^{-3/4+\varepsilon}]$ and $\xi \in [n^{-3/4+\varepsilon}, a]$.

Theorem 9.1. *If $\xi \in [n^{-3/4+\varepsilon}, a]$, for some $\varepsilon > 0$ and for sufficiently small a, then*

$$p^{(2n)}(2k,0) \overset{n}{\sim} \frac{4\,e^{n\varphi(\xi)} v(\xi)^3}{\pi\sqrt{a_2(\xi)}} \frac{G(z_o(\xi))}{z_o(\xi)}\, n^{-1/2} \int_0^{+\infty} e^{-\frac{\theta^2}{2}} \cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\theta^2\right) \mathrm{d}\theta$$

uniformly with respect to $\xi \in [n^{-3/4+\varepsilon}, a]$.

Proof. We apply Theorem 2.3, observing that the modulus of the integrand is majorized by

$$C(1+\theta)^3 \exp\left\{\frac{\theta^2}{2} - C\theta^3 - C\theta^4\right\},$$

for every $\theta \ge 0$, $\xi \in [n^{-3/4+\varepsilon}, a]$ and n. Moreover, the integrand converges pointwise, uniformly with respect to $\xi \in [n^{-3/4+\varepsilon}, a]$, to

$$e^{-i\frac{\pi}{3}} \exp\left\{e^{-\frac{2}{3}\pi i}\theta^2\right\}.$$

Negligibility of **(B)** is shown as usual. □

Theorem 9.2. *If $\xi \in [n^{-3/4}, n^{-3/4+\varepsilon}]$, for some $\varepsilon > 0$, then*

$$p^{(2n)}(2k,0) \overset{n}{\sim} \frac{2^{1/6} e^{n\varphi(\xi)}}{\pi\sqrt{3}} \frac{G(z_o(\xi))}{z_o(\xi)}\, I(n^{-1/2}\xi^{-2/3})\,\xi^{2/3} n^{-1/2},$$

uniformly with respect to $\xi \in [n^{-3/4}, n^{-3/4+\varepsilon}]$, where $I(t)$ is an integral function, defined as follows:

$$\begin{aligned} I(t) := \int_0^{+\infty} & \exp\left\{-\frac{\theta^2}{2} - \frac{2^{11/6} t}{6\sqrt{3}}\,\theta^3 - \frac{t^2}{9\cdot 2^{7/3}}\,\theta^4\right\} \cdot \\ & \cdot\left\{\sqrt{3}\cos\left(-\frac{\sqrt{3}\,\theta^2}{2} + \frac{\sqrt{3}\,t^2}{9\cdot 2^{7/3}}\,\theta^4\right) \cdot (1 + \sqrt{3}\,2^{-1/6} t\theta - (3\sqrt{6})^{-1} t^3\theta^3) + \right. \\ & \left. - \sin\left(-\frac{\sqrt{3}\,\theta^2}{2} + \frac{\sqrt{3}\,t^2}{9\cdot 2^{7/3}}\,\theta^4\right) \cdot (1 - \sqrt{3}\,2^{-1/6} t\theta - 2^{2/3} t^2\theta^2 - (3\sqrt{6})^{-1} t^3\theta^3)\right\} \cdot \\ & \cdot \left\{\mathbb{1}_{[0, \frac{2^{7/6}}{\sqrt{3}-1}\frac{\sqrt{3}}{t}]}(\theta) - \mathbb{1}_{[\frac{2^{7/6}}{\sqrt{3}-1}\frac{\sqrt{3}}{t}, +\infty)}(\theta)\right\}. \end{aligned}$$

Proof. The substantial difference with the previous case is that we cannot find a pointwise limit for the integrand, but only a pointwise asymptotic estimate. Hence we apply Theorem 2.4, exactly as in the proof of Theorem 8.2. □

10. Final remarks

First, we want to spend a few words to explain how one can obtain an estimate for the transition probabilities with odd time and space parameters.

From the results on the y-axis we obtain also $p^{(2n+1)}((0,2k+1),(0,0))$, since

$$p^{(2n+1)}((0,2k+1),(0,0)) = \frac{1}{2}\{p^{(2n)}(0,2k+2),(0,0)) + p^{(2n)}(0,2k),(0,0))\},$$

as we noted in Paragraph 4.

The estimate of $p^{(2n+1)}((2k+1,0),(0,0))$ requires further calculation, but it is not much different from what we did this far. We briefly point out what has to be done.

Let $G(z)$, $F(z)$ and $\Psi_\xi(z)$ be as in Paragraph 6, then

$$\begin{aligned} p^{(2n+1)}((2k+1,0),(0,0)) &= \frac{1}{2\pi i}\oint \frac{G(z)}{z}\frac{F(z)}{\sqrt{z}} \cdot \exp\{n\Psi_\xi(z)\}\,\mathrm{d}z = \\ &= \frac{1}{2\pi i}\oint \frac{G(z)}{z}\frac{1+\sqrt{1-z}-\sqrt{2}\sqrt{1-z+\sqrt{1-z}}}{z} \cdot \exp\{n\Psi_\xi(z)\}\,\mathrm{d}z. \end{aligned}$$

Thus, if $\xi \in [a, 1-c]$ the proof will differ from that of Theorem 6.2 because one should multiply and divide the integrand not by $\frac{G(z_o(\xi))}{z_o(\xi)}$ but by $\frac{G(z_o(\xi))\cdot F(z_o(\xi))}{z_o(\xi)^2}$. The rest of the proof is completely analogous to that of Theorem 6.2.

The same observation can be made in the case $\xi \geq n^{-3/4}$.

In the case $\xi \in [0, n^{-3/4}]$ one will need a decomposition (similar to the decomposition (5.10) of $G(z)$) of $F(z)\cdot\sqrt{z}$:

$$1+\sqrt{1-z}-\sqrt{2}\sqrt{1-z+\sqrt{1-z}} = 1+\sqrt{1-z}-\sqrt[4]{1-z}J(z)+\sqrt[4]{(1-z)^3}L(z),$$

where $J(z)$ and $L(z)$ are two holomorphic functions. Then

$$\begin{aligned} &G(z)\cdot(1+\sqrt{1-z}-\sqrt{2}\sqrt{1-z+\sqrt{1-z}}) = \\ &=(1-z)^{-1/4}H(z) + (1-z)^{1/4}(H(z)+K(z)) + (1-z)^{1/2}(J(z)\,K(z)+H(z)\,L(z)) + \\ &\quad + (1-z)^{3/4}K(z) + \textit{holomorphic functions}, \end{aligned}$$

and one can proceed as in Theorem 6.2.

Moreover, by symmetry, translation invariance and reversibility from our estimates of $p^{(n)}((k,0),(0,0))$ and $p^{(n)}((0,k),(0,0))$ (for $k>0$) one easily derives uniform estimates for $p^{(n)}((k,0),(k_1,0))$, $p^{(n)}((k_1,k),(k_1,0))$, $p^{(n)}((k_1,0),(k_1,k))$ for $k, k_1 \in \mathbb{Z}$.

We note that it is not possible to extend these estimates to uniform estimates for any starting and ending point, as we did in equationloc-lim2 for local estimates since the asymptotic we use (picked from [1] Paragraph 6) does not hold uniformly.

We now remark some particular features of the transition probabilities of the simple random walk on C_2 that appear after our calculations. We recall our estimates for ξ tending very fast to zero, and we stress the dependence on ξ. If $k/n = \xi \in [0, n^{-3/4-\varepsilon}]$ for some $\varepsilon > 0$, then

$$p^{(2n)}(x_k, o) \overset{n}{\sim} \begin{cases} \dfrac{\sqrt{2}\,e^{-n\xi^2}}{\Gamma(\frac{1}{4})}\,n^{-3/4}, & \text{if } x_k = (0,2k) \\ \dfrac{\sqrt{2}\,e^{-Cn\xi^{4/3}}}{\Gamma(\frac{1}{4})}\,n^{-3/4}, & \text{if } x_k = (2k,0) \end{cases} \tag{10.1}$$

uniformly with respect to $\xi \in [0, n^{-3/4-\varepsilon}]$.

It is worth noting that, even if both these results agree with the local limit estimate (substitute $\xi = 0$), they show a different dependence on ξ along the two axes. In particular this reflects in the impossibility of finding for C_2 an estimate of the type Jones found for the 2-dimensional Sierpiński graph ([7]), and Barlow and Bass for the graphical Sierpiński carpet ([8]). Recall that associated with a graph with polynomial growth there are three positive constants: the *spectral dimension* δ_s, the *fractal dimension* δ_f and the *walk dimension* δ_w. In typical cases, one has the so-called "Einstein relation": $\delta_s \cdot \delta_w = 2\delta_f$ (see Telcs [9], [10]). For C_2, $\delta_s = 3/2$ (see, for instance Cassi and Regina [4]) and $\delta_f = 2$. From the Einstein relation one would therefore expect $\delta_w = 8/3$. Here is Jones' estimate: for some positive constants c_1, c_2, c_3, c_4 for sufficiently large n and for all x, y in the two-dimensional Sierpiński graph,

$$c_1\, n^{-\delta_s/2} \exp\left(c_2\, n\xi^{\delta_w/(\delta_w-1)}\right) \leq p^{(n)}(x,y) \leq c_3\, n^{-\delta_s/2} \exp\left(c_4\, n\xi^{\delta_w/(\delta_w-1)}\right),$$

where $p^{(n)}(x,y)$ are the transition probabilities of the simple random walk and $\xi = d(x,y)/n$.

Now, equation (10.1) shows that if $x = (0, 2k)$ and $y = (0,0)$ one would get a "Jones"-type estimate with $\delta_w = 2$, while if $x = (0, 2k)$ and $y = (0,0)$ the estimate would do if $\delta_w = 4$. Observe that equation (10.1) implies that $p^{(2n)}((2k,0),(0,0))/p^{(2n)}((0,2k),(0,0))$ is not bounded below by a positive constant, as Jones' estimate would imply.

We finally observe that the method we used here to provide a uniform asymptotic estimate of transition probabilities has already been employed for homogeneous trees and free groups (see Lalley [11], [12] and [13]). One of the aims of this paper was also shedding new light onto this method. It seems that one could use the described technique for more general graphs and random walks (provided the knowledge of the Green function). As we noted, the major difficulty appears to be in the choice of a proper contour of integration, for which no recipe is known (we investigated only a restricted family of curves in our case).

A natural extension of our results could be the analogues for d-combs. However, technical difficulties increase notably already for $d = 3$, since

$$G_3(z) = \frac{3}{\sqrt{3(1-z^2) + 2\sqrt{1-z^2} + 2\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}}}.$$

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